Stochastic Calculus
LECTURE 1

This course is about developing a theory of calculus which is applicable to continuous time stochastic processes, e.g. Brownian motion. Why do we need a special theory?
Brownian motion is not differentiable.

Ordinary calculus	Stochastic calculus
Integral	Itô (stochastic) integral.
Derivative	Itô (stochastic) derivative.
ODEs.	SDEs

Example: Suppose that we have a gambler who repeatedly tosses a fair coin, betting £1 on getting a heads for each toss. Let

$$\xi_k = \begin{cases} +1, & \text{heads on } k^{th} \text{ toss} \\ -1, & \text{otherwise} \end{cases}$$

(ξ_k) iid Bernoulli$(1/2)$.
Let $X_n = \sum_{k=1}^{n} \xi_k$ is the net winnings of the gambler. Note that (X_n) is a simple random walk on $\mathbb{Z}$, $X_0 = 0$, hence is a martingale (MG) w.r.t. $\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$. Suppose that at the n^{th} toss, bet £H_n on heads. Then:

$$(H \cdot X)_n = \sum_{k=1}^{n} H_k \cdot (X_k - X_{k-1})$$

gives the net winnings after n tosses.
Assume that (H_n) is a deterministic sequence.
Claim: $H \cdot X$ is an $\mathcal{F}_n$-MG.
a) integrable ✓
b) adapted ✓
c) $\mathbb{E}[(H \cdot X)_{n+1} - (H \cdot X)_n \mid \mathcal{F}_n]$
$= H_{n+1} \cdot \mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n] = 0.$

More generally, the same is true if we take H_{n+1} is $\mathcal{F}_n$-measurable [and integrable]. This is called a previsible process. As before, $H \cdot X$ gives the net winnings of the gambler. This is called a MG transform.

Goal for first part of the course:
extend this reasoning to define

$$(H \cdot X)_t = \int_0^t H_s dX_s \quad (*)$$

where H is previsible, X is a continuous MG (e.g. Brownian Motion). Cannot use the Lebesgue-Stieltjes integrals to define $(*)$ since this requires X to have finite variation and the only continuous martingales which have finite variation are constant.

Strategy to define the Itô integral:

$$\text{Set } (H \cdot X)_t \text{ "="} \lim_{\varepsilon \to 0} \sum_{k=1}^{\lfloor t/\varepsilon \rfloor} H_{k\varepsilon} (X_{(k+1)\varepsilon} - X_{k \cdot \varepsilon})$$

We need to be careful about the type of limit since X in general will be rough (not differentiable) like Brownian motion.
To get convergence, we need to take advantage of cancellations.
For example, if X is a Brownian motion and H is a deterministic, and continuous process.
We have

$$\mathbb{E}\left[\left(\sum_{k=0}^{\lfloor t/\varepsilon \rfloor} H_{k\varepsilon} \cdot (X_{(k+1)\varepsilon} - X_{k\varepsilon})\right)^2\right] =$$

$$\mathbb{E}\Big[\sum_{k=0}^{\lfloor t/\varepsilon \rfloor} H_{k\varepsilon}^2 \cdot (X_{(k+1)\varepsilon} - X_{k\varepsilon})^2 + \underbrace{\sum_{j \neq k} H_{k\varepsilon} H_{j\varepsilon} (X_{(k+1)\varepsilon} - X_{k\varepsilon})(X_{(j+1)\varepsilon} - X_{j\varepsilon})}_{=0}\Big]$$

$$= \mathbb{E}\Big[\sum_{k=0}^{\lfloor t/\varepsilon \rfloor} H_{k\varepsilon}^2 \cdot (X_{(k+1)\varepsilon} - X_{k\varepsilon})^2\Big]$$

$$= \sum_{k=0}^{\lfloor t/\varepsilon \rfloor} H_{k\varepsilon}^2 \cdot \varepsilon \longrightarrow \int_0^t H_s^2 \, ds \text{ as } \varepsilon \to 0.$$

Cancellations come from MG orthogonality and are what makes it possible to define the Itô integral.

Next, learn about properties of the integral:
- Stochastic analogue of the chain rule.
- Stochastic analogue of integration by parts

Formulas look like those in regular calculus but with an extra term to reflect that X is rough (quadratic variation).

$$Y_t = \int_0^t H_s dX_s \iff dY_t = H_t dX_t$$

Itô's formula: how to write $df(Y_t)$ in terms of dY_t for $f \in C^2$.
Many applications, for example the Dubins-Schwarz theorem:

any continuous MG is a time-change of Brownian motion.

Next, look at stochastic differential equations (SDEs), i.e.

$$dX_t = b(t, X_t)dt + \sigma(t, X_t) dB_t,$$

where b, σ are "nice" and B is a Brownian motion.
For $\sigma \equiv 0$, just an ODE. For $\sigma \not\equiv 0$, corresponds to adding noise which depends on time and the state of the system.

Last part of the course: diffusion processes and how they are related to SDEs, and how they can be used to solve PDEs involving 2^{nd} order elliptic equations (e.g. Δ).

Next time: preliminaries (cadlag processes, function of finite variation, integral against a function / process of finite variation).

Stochastic Calculus
LECTURE 2

Recall that $a : [0,\infty) \to \mathbb{R}$ is cadlag if it is right-continuous and has left hand limits: $\lim_{y\to x^-} a(y)$ exists, $\lim_{y\to x^+} a(y) = a(x)$.

Let $a(x-)$ be the left-hand limit, $\Delta a(x) = a(x) - a(x-)$.

Suppose that a is non-decreasing, cadlag, $a(0)=0$. Then there exists a unique Borel measure da on $[0,\infty)$ such that
$da((s,t]) = a(t) - a(s)$ $\forall\, 0 \le s < t$.

For f measurable and integrable, then the Lebesgue-Stieltjes integral $f \cdot a$ is defined by
$$(f\cdot a)_t = \int_{(0,t]} f(s)\, da(s) \quad \forall t \ge 0.$$

Then $f\cdot a$ is a right-continuous function. Moreover, if a is continuous, then $f \cdot a$ is continuous and so we can write
$$\int_{(0,t]} f(s)\, da(s) = \int_0^t f(s)\, da(s).$$

Want to integrate against more functions. Suppose that a^+, a^- are functions satisfying the same conditions as before and set $a = a^+ - a^-$. Define $(f\cdot a)(t) = (f\cdot a^+)(t) - (f \cdot a^-)(t)$ for all f measurable so that both terms on the RHS are finite. This class of functions
(= differences of cadlag non-decreasing functions) coincide with the cadlag functions with <u>finite variation</u>.

<u>Definition</u>: Let $a : [0,\infty) \to \mathbb{R}$ be cadlag. For each $n \in \mathbb{N}$, $t \ge 0$, let

(✱) $$v^n(t) = \sum_{k=0}^{\lceil 2^n t\rceil - 1} |a((k+1)\cdot 2^{-n}) - a(k\cdot 2^{-n})|$$

Then the limit $\lim_{n\to\infty} v^n(t) = v(t)$ exists and is called the <u>total variation</u> of a on $[0,t]$.

If $v(t) < \infty$, then we say that a has <u>finite variation</u> on $[0,t]$. If a has finite variation on $[0,t]$ $\forall t \ge 0$ we say that a is a function of finite variation.

To see that $\lim_{n\to\infty} v^n(t)$ exists, fix $t > 0$ and let $t_n^+ = 2^{-n}\lceil 2^n t\rceil$ so that $t_n^+ \ge t \ge t_n^-$ $\forall n$
$t_n^- = 2^{-n}(\lceil 2^n t\rceil - 1)$
and $$v^n(t) = \sum_{k=0}^{2^n t_n^- - 1} |a((k+1)\cdot 2^{-n}) - a(k\cdot 2^{-n})| + |a(t_n^+) - a(t_n^-)|$$

Δ-inequality $\Rightarrow$ first-term is non-decreasing in n. Cadlag property $\Rightarrow$ second term $\to |\Delta a(t)|$

$\Rightarrow$ $v^n(t)$ converges as $n \to \infty$.

<u>Lemma</u>: Let a be a cadlag function of finite variation. Then v is also cadlag of finite variation with $\Delta v(t) = |\Delta a(t)|$ $\forall t \ge 0$ and v is non-decreasing. In particular, if a is continuous, then so is v.

<u>Proof</u>: Ex. Sheet 1.

<u>Proposition</u> A cadlag function can be written as a difference of two right-continuous non-decreasing functions if and only if it has finite variation.

<u>Proof</u>: Assume that $a = a^+ - a^-$ are cadlag, non-decreasing. NTS: a has finite variation.
Note, $|a(t) - a(s)| \le (a^+(t) - a^+(s)) + (a^-(t) - a^-(s))$ $\forall 0 \le s < t$

Plug this into (✱) and use that the sum telescopes for monotone functions to get that
$v^n(t) \le (a^+(t_n^+) - a^+(0)) + (a^-(t_n^+) - a^-(0))$.
Since $a^{+/-}$ are right continuous,
RHS $\to (a^+(t) - a^+(0)) + (a^-(t) - a^-(0))$
$\Longrightarrow$ a has finite variation.

Now the reverse direction. Assume that $v(t) < \infty$ $\forall t \ge 0$. Set $a^+ = \frac{1}{2}(v + a)$, $a^- = \frac{1}{2}(v - a)$. Then $a = a^+ - a^-$ and $a^{+/-}$ are cadlag since v, a are cadlag. NTS: $a^{+/-}$ are non-decreasing. Fix $0 \le s < t$, define $t_n^{+/-}$ as before and $s_n^{+/-}$ analogously.
Then: $$a^+(t) - a^+(s) = \lim_{n\to\infty} \frac{1}{2}\left(v^n(t) - v^n(s) + a(t_n^+) - a(s_n^+)\right)$$
$$= \lim_{n\to\infty} \frac{1}{2}\Big[\sum_{k=2^n s_n^+}^{2^n t_n^- - 1} \underbrace{\big(|a((k+1)\cdot 2^{-n}) - a(k\cdot 2^{-n})| + (a((k+1)\cdot 2^{-n}) - a(k\cdot 2^{-n}))\big)}_{\ge 0} + \underbrace{|a(t_n^+) - a(t_n^-)| + (a(t_n^+) - a(t_n^-))}_{\ge 0}\Big]$$
≥ 0.

Same argument works for a^- □.

<u>Random integrands</u>: Now discuss integration against random functions of finite variation.
$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ filtered probability space.
Recall $X : \Omega \times [0,\infty) \to \mathbb{R}$ is <u>adapted</u> to $(\mathcal{F}_t)_{t \ge 0}$ if $X_t = X(\cdot, t)$ is $\mathcal{F}_t$-measurable $\forall t \ge 0$.
X is <u>cadlag</u> if $X(\omega, \cdot)$ is cadlag $\forall \omega \in \Omega$.

<u>Definition</u>: Given a cadlag, adapted process $A : \Omega \times [0,\infty) \to \mathbb{R}$, its <u>total variation process</u> $V : \Omega \times [0,\infty) \to \mathbb{R}$ is pathwise by setting $V(\omega, \cdot)$ to be the total variation of $A(\omega, \cdot)$.

<u>Lemma</u>: if A is cadlag, adapted, and of finite variation $\Rightarrow$ V is cadlag, adapted, non-decreasing.

<u>Proof</u>: NTS V is adapted.
For $t \ge 0$, $t_n^- = 2^{-n}(\lceil 2^n t\rceil - 1)$
$$\tilde{V}_t^n = \sum_{k=0}^{2^n t_n^- - 1} |A_{(k+1)\cdot 2^{-n}} - A_{k\cdot 2^{-n}}|$$
$\tilde{V}^n$ is adapted $\forall n$ since $t_n^- \le t$.
$V_t = \lim_{n\to\infty} \tilde{V}_t^n + |\Delta A(t)|$ $\Rightarrow$ V_t is $\mathcal{F}_t$-measurable.

Stochastic Calculus
Lecture 3

Today: Look at class of functions so that the integral is adapted.

Recall from the intro: a discrete time process (H_n) is previsible wrt $(\mathcal{F}_n)$ if H_{n+1} is $\mathcal{F}_n$-measurable $\forall n$.

Definition: The previsible σ-algebra $\mathcal{P}$ on $\Omega \times (0,\infty)$ is the σ-algebra which is generated by sets of the form $E \times (s,t]$ where $E \in \mathcal{F}_s$, $s<t$. A process $H: \Omega \times (0,\infty) \to \mathbb{R}$ is previsible if it is measurable wrt $\mathcal{P}$.

Examples: ① $H(\omega,t) = Z(\omega)\cdot \mathbb{1}_{(t_1,t_2]}(t)$, $t_1 < t_2$, Z $\mathcal{F}_{t_1}$-meas.

② $H(\omega,t) = \sum_{k=0}^{n-1} Z_k(\omega)\cdot \mathbb{1}_{(t_k,t_{k+1}]}(t)$ for $0 = t_0 < \dots < t_n$ and Z_k is $\mathcal{F}_{t_k}$-meas.

"Simple process"; will be important for the construction of the Itô integral.

Remarks: Simple processes are left-continuous and adapted. It turns out that $\mathcal{P}$ is the smallest σ-algebra on $\Omega \times (0,\infty)$ so that all left-continuous processes are measurable.
In general, cadlag processes are not previsible but their left-continuous modification is.

Proposition: Let X be a cadlag, adapted process and let $H_t = X_{t-}$, $t \geq 0$. Then H is previsible.

Proof: Since X is cadlag and adapted, it is clear that H is left-continuous and adapted. For each n, set
$$H_t^n = \sum_{k=0}^{\infty} H_{k\cdot 2^{-n}} \cdot \mathbb{1}_{(k\cdot 2^{-n}, (k+1)\cdot 2^{-n}]}(t)$$
Then H_t^n is previsible $\forall n$ and since H is a left-continuous process,
$$\lim_{n\to\infty} H_t^n = H_t \quad \forall t \Rightarrow H \text{ is previsible as a}$$
limit of previsible processes. $\square$

Remark: Prop'n $\Longrightarrow$ continuous, adapted processes are previsible.

Proposition: If H is previsible, then H_t is measurable wrt. $\mathcal{F}_{t-} = \sigma(\mathcal{F}_s : s<t)$ $\forall t \geq 0$.

Proof: Ex. Sheet 1.

Remark: The Poisson process (N_t) is not previsible since N_t is not $\mathcal{F}_{t-}$-meas. where $(\mathcal{F}_t)$ is the natural filtration.

Now going to show that integrating a previsible process against a cadlag process which is adapted and has finite variation yields an adapted cadlag process of finite variation.

Theorem: Let $A: \Omega \times (0,\infty) \to \mathbb{R}$ be a cadlag process which is adapted and has finite variation V. Let H be a previsible process with

(1) $\int_{(0,t]} |H(\omega,s)|\, dV(s) < \infty \quad \forall t \geq 0, \omega \in \Omega$.

Then the process $H\circ A: \Omega \times (0,\infty) \to \mathbb{R}$ given by

(2) $(H\circ A)(\omega,t) = \int_{(0,t]} H(\omega,s)\, dA(\omega,s)$, $(H\circ A)(\omega,0) = 0$.

is cadlag, adapted and has finite variation.

Proof: The integral in (2) is well-defined due to (1). Indeed, let $H^{+/-} = \max\{\pm H, 0\}$, $A^{+/-} = 1/2\,(V +/- A)$. Then $H = H^+ - H^-$ and $A = A^+ - A^-$ and
$$H\circ A = (H^+ - H^-)\cdot(A^+ - A^-)$$
$$= H^+\circ A^+ + H^-\circ A^- - H^+\cdot A^- - H^-\cdot A^+$$
and all terms on RHS are finite by (1).

NTS: $H\cdot A$ is ① cadlag, ② adapted, ③ finite variation.

Step ①: Note that $\mathbb{1}_{(0,s]} \to \mathbb{1}_{(0,t]}$ as $s \downarrow t$
$\mathbb{1}_{(0,s]} \to \mathbb{1}_{(0,t)}$ as $s \uparrow t$.

By definition, $(H\circ A)_t = \int H_s \cdot \mathbb{1}(s \in [0,t])\, dA_s$

So, $(H\cdot A)_t = \int H_s \cdot \lim_{r \downarrow t} \mathbb{1}(s \in (0,r])\, dA_s$

(DCT) $\hookrightarrow$ $= \lim_{r\downarrow t} \int H_s \cdot \mathbb{1}(s \in (0,r])\, dA_s$

$= \lim_{r \downarrow t} (H\circ A)_r \Rightarrow$ is right continuous

Analogous argument $\Rightarrow$ $H\cdot A$ has left-limits, hence cadlag. Also, $\Delta(H\circ A)_t = \int H_s \cdot \mathbb{1}(s=t)\, dA_s$
$= H_t\, \Delta A_t$

Step 2: "Monotone class" style argument.
Suppose $H = \mathbb{1}_{B\times(s,u]}$, $B \in \mathcal{F}_s$, $s<u$. Then $(H\circ A)_t = \mathbb{1}_B(A_{t\wedge u} - A_{t\wedge s})$, which is $\mathcal{F}_t$-measurable. Let $\mathcal{A} = \{C \in \mathcal{P} : \mathbb{1}_C \cdot A \text{ is adapted}\}$.

WTS: $\mathcal{A} = \mathcal{P}$.

Let $\Pi = \{B \times (s,u],\ B \in \mathcal{F}_s,\ s<u\}$. We have shown $\Pi \subseteq \mathcal{A}$, know that Π is a π-system generating $\mathcal{P}$. Not difficult to see that $\mathcal{A}$ is a d-system and by Dynkin's lemma
$\Longrightarrow \mathcal{P} = \sigma(\Pi) \subseteq \mathcal{A} \subseteq \mathcal{P} \Rightarrow \mathcal{A} = \mathcal{P}$.

Now suppose that $H \geq 0$ is previsible.
Set $H^n = (2^{-n}\lfloor 2^n H \rfloor) \wedge n$
$$= \sum_{k=0}^{n2^n-1} 2^{-n}\cdot k\cdot \underbrace{\mathbb{1}(H \in [2^{-n}k, (k+1)2^{-n}))}_{\in \mathcal{P}} + n\cdot \underbrace{\mathbb{1}(H \geq n)}_{\in \mathcal{P}}$$

$\Rightarrow$ H^n is a finite linear combination of functions of the form $\mathbb{1}_C$, $C \in \mathcal{P}$
$\Rightarrow$ $(H^n\cdot A)_t$ is $\mathcal{F}_t$-meas. $\forall t$. Monotone conv. thm: $(H^n\cdot A)_t \to (H\cdot A)_t$ as $n\to\infty$.
For general H, write $H = H^+ - H^-$, $H^{+/-} = \max(\pm H, 0)$ and use that $(H\cdot A)_t = \underbrace{(H^+\cdot A)_t - (H^-\cdot A)_t}_{\text{both } \mathcal{F}_t\text{-meas.}}$

Step 3: Finite variation
$$H\cdot A = (H^+ - H^-)\cdot(A^+ - A^-)$$
$$= (H^+\cdot A^+ + H^-\cdot A^-) - (H^-\cdot A^+ + H^+\cdot A^-)$$
is a difference of non-decreasing functions

$\square$

Next: integrating against MGs.

STOCHASTIC CALCULUS

LECTURE 4

Local Martingales:

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space.

Definition: Say that $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions if:

- $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets.
- $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous: $\mathcal{F}_t = \mathcal{F}_{t^+} = \bigcap_{s>t} \mathcal{F}_s \; \forall t \geq 0$.

Throughout, assume that $(\mathcal{F}_t)$ satisfies the usual conditions. Recall that an integrable adapted process X is an $(\mathcal{F}_t)$ MG if $\mathbb{E}[X_t | \mathcal{F}_s] = X_s$ a.s.

sup MG if $\mathbb{E}[X_t | \mathcal{F}_s] \leq X_s$ a.s.

sub MG if $\mathbb{E}[X_t | \mathcal{F}_s] \geq X_s$ a.s.

for all $0 \leq s < t$.

A random variable T is called a stopping time if $\{T \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. If X is cadlag and adapted to $(\mathcal{F}_t)$ and we set $\mathcal{F}_T = \{E \in \mathcal{F} : E \cap \{t \leq T\} \in \mathcal{F}_t \; \forall t \geq 0\}$ then X_T is an $\mathcal{F}_T$-measurable random variable. If X is a martingale then $X_t^T = X_{t\wedge T}$ is also a martingale

Theorem (OST) Let X be an adapted, cadlag and integrable process process. Then the following are equivalent:

1) X is a MG.
2) $X^T = (X_{t\wedge T})_{t\geq 0}$ is a MG $\forall$ stopping time T.
3) for all bounded stopping times $S \leq T$, then $\mathbb{E}[X_T | \mathcal{F}_S] = X_S$ a.s.
4) for all bounded stopping times T, we have that $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

Definition: A cadlag adapted process X is called a local martingale (MG) if there exists a sequence $(T_n)_{n\geq 1}$ of stopping times with $T_n \uparrow \infty$ a.s. (non-decreasing) such that the stopped process X^{T_n} is a martingale for all $n \geq 1$. In this case, we say that (T_n) reduces X.

Note that a MG is a local martingale as any deterministic sequence $T_n \uparrow \infty$ will reduce it.

Example: let B be a standard Brownian motion in $\mathbb{R}^3$. Let $M_t = \frac{1}{|B_t|}$. Example Sheet #4 in Advanced Probability:

i) $(M_t)_{t\geq 1}$ is L^2 bounded: $\sup_{t\geq 1} \mathbb{E}[M_t^2] < \infty$.

ii) $\mathbb{E}[M_t] \to 0$ as $t \to \infty$.

iii) M is a supermartingale.

M cannot be a MG otherwise its expectation would vanish by ii) but this cannot be true since $M_t > 0$ a.s.

For each $n \geq 1$, set:

$$T_n = \inf\{t \geq 1 : |B_t| < 1/n\} = \inf\{t \geq 1 : |M_t| > n\}.$$

WTS:
1) $(M_t^{T_n})_{t\geq 1}$ is a martingale $\forall n$.
2) $T_n \uparrow \infty$ as $n \to \infty$ a.s.

Note:
$n \leq M_1 \Rightarrow T_n = 1$.
$n > M_1 \Rightarrow T_n > 1$.

Since $|B_t|$ cannot hit $1/n$ before hitting $1/{n+1}$, have that T_n is non-decreasing.

Advanced Probability: $f \in C_b^2(\mathbb{R}^3)$,

$$f(B_t) - f(B_0) - 1/2 \int_0^t \Delta f(B_s) ds \text{ is a MG.}$$

Note that $f(x) = \frac{1}{|x|}$ is a harmonic function in $\mathbb{R}^3 \setminus \{0\}$. Let $(f_n)_{n\geq 1}$ a sequence of $C_b^2(\mathbb{R}^3)$ with $f_n(x) = f(x) = 1/|x|$ on $\{|x| \geq 1/n\}$. If $0 < |B_1| < 1/n$, then $T_n = 1$ and so $M_t^{T_n} = M_1$ is a martingale. Since $B_1 \neq 0$ a.s., we have that $|B_1| > 1/n$ for all n sufficiently large enough in which case

$$f(B_{t\wedge T_n}) = f^n(B_{t\wedge T_n}), \; \forall t \geq 1.$$

Thus:
$$M_{t\wedge T_n} = f(B_{t\wedge T_n}) - f(B_1) + f(B_1)$$
$$= \left[f(B_{t\wedge T_n}) - f(B_1) - \frac{1}{2}\int_1^{t\wedge T_n} \Delta f(B_s) ds\right] + f(B_1)$$
$$\text{MG} \Big\{ = \left[f^n(B_{t\wedge T_n}) - f^n(B_1) - \frac{1}{2}\int_1^{t\wedge T_n} \Delta f^n(B_s) ds\right] + f^n(B_1)$$

$\Rightarrow M^{T_n} = (M_{t\wedge T_n})_{t\geq 1}$ is a MG.

NTS: $T_n \uparrow \infty$ as $n \to \infty$. As $T_n \leq T_{n+1}$,

NTS: $\lim_{n\to\infty} T_n = \infty$ a.s.

For each R let $S_R = \inf\{t \geq 1 : |B_t| > R\} = \inf\{t \geq 1 : M_t < 1/R\}$

Then $S_R \to \infty$ as $R \to \infty$.

$$\mathbb{P}(\lim_{n\to\infty} T_n < \infty) \leq \mathbb{P}(\exists R : T_n < S_R \; \forall n) = \lim_{R\to\infty} \lim_{n\to\infty} \mathbb{P}(T_n < S_R).$$

OST $\Rightarrow \mathbb{E}[M_{T_n \wedge S_R}] = \mathbb{E}[M_1] = N \in (0, \infty)$.

LHS: $\quad n \cdot \mathbb{P}(T_n < S_R) + 1/R \, \mathbb{P}(S_R \leq T_n)$

$\mathbb{P}(S_R \leq T_n) = 1 - \mathbb{P}(T_n < S_R)$

$$\Rightarrow \mathbb{P}(T_n < S_R) = \frac{N - 1/R}{n - 1/R} \to 0 \text{ as } n \to \infty.$$

Therefore, $(M_t)_{t\geq 0}$, non-negative local MG but not a MG, a super MG and is L^2 bded.

First two properties $\Longrightarrow$ sup MG

Proposition: If X is a loc. MG, $X_t \geq 0 \; \forall t \geq 0$, then X is a sup MG.

Proof: Suppose that (T_n) is a reducing sequence. Then for any $s \leq t$, we know that

$$\mathbb{E}[X_t | \mathcal{F}_s] = \mathbb{E}[\lim_{n\to\infty} X_{t\wedge T_n} | \mathcal{F}_s]$$
$$\leq \liminf_{n\to\infty} \mathbb{E}[X_{t\wedge T_n} | \mathcal{F}_s] \quad \text{(Fatou)}$$
$$= \liminf_{n\to\infty} X_{s\wedge T_n} = X_s \text{ a.s.} \qquad \square$$

Often work with loc. MGs instead of MGs so don't worry about integrability.

LECTURE 5

Last time: local MGs

Today: (1) when is a local MG a MG.
(2) continuous local MG with finite variation is constant.

Definition: A collection $\mathcal{X}$ of random variables is called *uniformly integrable* (UI) if
$$\sup_{X \in \mathcal{X}} \mathbb{E}[|X| \cdot \mathbb{1}(|X| > \lambda)] \to 0 \quad \text{as } \lambda \to \infty$$

Examples of UI families:

(1) Uniformly bounded random variables. $\exists C > 0$ deterministic so that $|X| \le C$ for all $X \in \mathcal{X}$.

(2) L^p bounded for $p > 1$:
$$\sup_{X \in \mathcal{X}} \mathbb{E}[|X|^p] < \infty$$

(3) $\exists Y$ integrable so that $|X| \le Y \ \forall X \in \mathcal{X}$.

Lemma: Suppose that $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Then $\mathcal{X} = \{\mathbb{E}[X|\mathcal{G}] : \mathcal{G} \text{ is a sub-}\sigma\text{-algebra of } \mathcal{F}\}$ is also a UI family.

Proof: Example Sheet #1.

Proposition: The following are equivalent:

i) X is a MG.

ii) X is a local MG and for all $t \ge 0$ the family $\mathcal{X}_t = \{X_T : T \text{ is a stopping time with } T \le t\}$ is UI.

Proof: i) $\Rightarrow$ ii). Suppose X is a MG. By OST, if T is a stopping time with $T \le t$, then $\mathbb{E}[X_t | \mathcal{F}_T] = X_T \Rightarrow \mathcal{X}_t$ is UI.

ii) $\Rightarrow$ i) Suppose that X is a local MG and $\mathcal{X}_t$ is UI for all $t \ge 0$. To show that X is a MG, by OST, it suffices to show that for all bdd stopping times T, we have $\mathbb{E}[X_T] = \mathbb{E}[X_0]$. Let (T_n) be a reducing sequence for X and let $T \le t$ be a stopping time.
$$\mathbb{E}[X_0] = \mathbb{E}[X_0^{T_n}] = \mathbb{E}[X_T^{T_n}] \overset{OST}{=} \mathbb{E}[X_{T \wedge T_n}]$$
Def^{n} of X^{T_n}

Since $\{X_{T \wedge T_n} : n \ge 0\}$ is UI and $X_{T \wedge T_n} \to X_T$ a.s.

Advanced Probability $\Rightarrow X_{T \wedge T_n} \to X_T$ in L^1 as $n \to \infty$. Therefore $\mathbb{E}[X_{T \wedge T_n}] \to \mathbb{E}[X_T]$ as $n \to \infty$.
Hence $\mathbb{E}[X_0] = \mathbb{E}[X_T]$.
OST $\Rightarrow$ X is a MG. $\square$

Corollary: A bounded local MG is a MG. More generally, if X is a local martingale and $\exists Y$ integrable such that $|X_t| \le Y$ $\forall t \ge 0$, then X is a MG.

Theorem: let X be a continuous local MG with $X_0 = 0$. If X has finite variation, then $X \equiv 0$ a.s.

Proof: Let V be the total variation process for X. Then $V_0 = 0$ continuous, adapted and non-decreasing. Let $T_n = \inf\{t \ge 0 : V_t = n\}$ for $n \in \mathbb{N}$. Then $T_n \uparrow \infty$ as $n \to \infty$ since X has finite variation. Moreover,
$$|X_t^{T_n}| = |X_{t \wedge T_n}| \le V_{t \wedge T_n} \le n.$$
$\Rightarrow X^{T_n}$ is a bounded local MG
$\Rightarrow X^{T_n}$ is a MG.

To prove that $X \equiv 0$, NTS: $X^{T_n} \equiv 0$ $\forall n$ [$T_n \uparrow \infty$ as $n \to \infty$]. Fix $n \in \mathbb{N}$, let $Y = X^{T_n}$. Y is a continuous bounded martingale with $Y_0 = 0$. To prove that $Y \equiv 0$, suffices to show that $\mathbb{E}[Y_t^2] = 0$ $\forall t \ge 0$. [this implies that $Y_t = 0$ $\forall t \ge 0, t \in \mathbb{Q}$ a.s., so $Y \equiv 0$ by continuity]. Fix $t \ge 0$, $N \in \mathbb{N}$, $t_k = \frac{k}{N} t$ for $0 \le k \le N$.
$$\mathbb{E}[Y_t^2] = \mathbb{E}\Big[\sum_{k=0}^{N-1} (Y_{t_{k+1}}^2 - Y_{t_k}^2)\Big]$$
$$= \mathbb{E}\Big[\sum_{k=0}^{N-1} (Y_{t_{k+1}} - Y_{t_k})^2\Big]$$
(MG orthogonality)
$$\le \underbrace{\mathbb{E}\Big[\overbrace{\max_{0 \le k \le N-1} |Y_{t_{k+1}} - Y_{t_k}|}^{\le V_{t \wedge T_n} \le n} \times \underbrace{\sum_{k=0}^{N-1} |Y_{t_{k+1}} - Y_{t_k}|}_{\le V_{t \wedge T_n}}\Big]}_{\le n^2}$$

Since Y is continuous, $\lim_{N \to \infty} \left(\max_{0 \le k \le N-1} |Y_{t_{k+1}} - Y_{t_k}| \right) = 0$ a.s.

Bounded convergence $\Rightarrow \mathbb{E}[Y_t^2] = 0$. $\square$

Remarks: i) Proof requires continuity, in particular not true without continuity.
ii) Theorem $\Rightarrow$ Brownian motion has infinite variation, so cannot use Lebesgue-Stieltjes integral to define the integral against a BM.

For continuous local MG, there is always an explicit way of choosing the reducing sequence.

Proposition: Let X be a continuous local MG with $X_0 = 0$. Then $T_n = \inf\{t \ge 0 : |X_t| = n\}$ reduces X.

Proof: *Step 1* T_n is a stopping time.
$$\{T_n \le t\} = \{\sup_{0 \le s \le t} |X_s| \ge n\}$$
$$= \bigcap_{k=1}^{\infty} \bigcup_{s \le t, s \in \mathbb{Q}} \underbrace{\{|X_s| > n - 1/k\}}_{\in \mathcal{F}_t}.$$

Step 2: $T_n \uparrow \infty$ as $n \to \infty$.
Since $\sup_{0 \le s \le t} |X_s(\omega)| < \infty \Rightarrow \exists n(\omega, t) \in \mathbb{N}$ s.t. $n(\omega, t) \ge \sup_{0 \le s \le t} |X_s(\omega)|$.
$$\Rightarrow n \ge n(\omega, t) \Rightarrow T_n(\omega) > t \Rightarrow T_n(\omega) \to \infty \text{ as } n \to \infty.$$

Step 3: (T_n) reduces X. Let (T_m^*) be a reducing sequence (exists since X is a local MG). Then $X^{T_m^*}$ is a MG $\forall m$. NTS: X^{T_n} is a MG.
OST $\Rightarrow X^{T_n \wedge T_m^*}$ is a MG $\forall m$
$\Rightarrow X^{T_n}$ is a local martingale with reducing sequence (T_m^*).
Since X^{T_n} is in addition bounded, it is a MG. $\Rightarrow (T_n)$ reduces X. $\square$

Next time: stochastic integral.

LECTURE 6

The Stochastic Integral

Goal: be able to integrate against a continuous local MG.
How does one construct an integral (Riemann, Lebesgue)?
An integral is a linear map
$I: X \longrightarrow Y$ where X, Y are normed vector spaces.

Steps: ① Define it on a dense set $D \subseteq X$

② Show that it is a continuous linear map:
$\exists C \geq 0$ s.t. $\|I(f)\|_Y \leq C \cdot \|f\|_X \; \forall f \in D$.
$\Rightarrow I$ extends by continuity to X.

Need to $\underbrace{\text{① specify } D, X, Y}_{\text{simple processes, quadratic variation}}$, $\underbrace{\text{prove ②.}}_{\text{Itô isometry}}$

Theorem: Let X be a cadlag, L^2-bounded MG ($\sup_{t \geq 0} \mathbb{E}[X_t^2] < \infty$). There exists X_∞ s.t. $X_t \to X_\infty$ a.s. and in L^2 and $\mathbb{E}[X_\infty | \mathcal{F}_t] = X_t \; \forall t \geq 0$ (X_∞ is called the "final value" of X).

Proposition: (Doob's L^2-inequality) Let X be a cadlag, L^2 bounded MG. Then
$\mathbb{E}[\sup_{t \geq 0} |X_t|^2] \leq 4\, \mathbb{E}[X_\infty^2]$

Define: $\mathcal{M}^2 = \{L^2 \text{ bounded, cadlag MGs}\}$
$\mathcal{M}^2_c = \{L^2 \text{ bounded, cont. MGs}\}$
$\mathcal{M}^2_{c,loc} = \{L^2 \text{ bounded, continuous local MGs}\}$

Definition: A process $H: \Omega \times (0, \infty) \to \mathbb{R}$ is called a simple process if it is of the form $H(\omega, t) = \sum_{k=0}^{n-1} Z_k(\omega) \cdot \mathbb{1}_{(t_k, t_{k+1}]}(t)$ for $n \in \mathbb{N}$, $0 = t_0 < t_1 < \dots < t_n$, Z_k bounded, $\mathcal{F}_{t_k}$-measurable random variable $\forall k$.
Let S be the set of simple processes.

- Define $(H \cdot M)_t$ for $H \in S$, $M \in \mathcal{M}^2$
- Extend the integral to more general integrands $[M \in \mathcal{M}^2_c]$

Integrating a simple process: Suppose that $H_t = \sum_{k=0}^{n-1} Z_k \cdot \mathbb{1}_{(t_k, t_{k+1}]}(t)$ is a simple process, $M \in \mathcal{M}$. Set:
$(H \cdot M)_t = \sum_{k=0}^{n-1} Z_k \cdot (M_{t \wedge t_{k+1}} - M_{t \wedge t_k})$

Proposition: If $H \in S$, $M \in \mathcal{M}^2$, then $H \cdot M \in \mathcal{M}^2$.
Moreover, $\mathbb{E}[(H \cdot M)_\infty^2] = \sum_{k=0}^{n-1} \mathbb{E}[Z_k^2 (M_{t_{k+1}} - M_{t_k})^2]$
$\leq 4 \cdot \|H\|_\infty^2 \, \mathbb{E}[(M_\infty - M_0)^2]$

Proof: Step 1 $H \cdot M$ is a MG. Suppose that $t_k \leq s < t \leq t_{k+1}$. Then we have that
$(H \cdot M)_t - (H \cdot M)_s = Z_k \cdot (M_t - M_s)$ so that
$\mathbb{E}[(H \cdot M)_t - (H \cdot M)_s | \mathcal{F}_s] = Z_k \cdot \mathbb{E}[M_t - M_s | \mathcal{F}_s] = 0$
since Z_k is $\mathcal{F}_s$-meas., $M \in \mathcal{M}^2$.
Suppose that $0 \leq t_j \leq s \leq t_{j+1} \leq t_k \leq t \leq t_{k+1}$
$\mathbb{E}[(H \cdot M)_t - (H \cdot M)_s | \mathcal{F}_s]$
$= \mathbb{E}[\sum_{i=0}^{k-1} Z_i \cdot (M_{t_{i+1}} - M_{t_i}) + Z_k \cdot (M_t - M_{t_k}) - (\sum_{i=0}^{j-1} Z_i (M_{t_{i+1}} - M_{t_i}) + Z_j (M_s - M_{t_j})) | \mathcal{F}_s]$
$= \sum_{i=j+1}^{k-1} \mathbb{E}[Z_i \cdot (M_{t_{i+1}} - M_{t_i}) | \mathcal{F}_s]$
$+ \mathbb{E}[Z_j (M_{t_{j+1}} - M_s) | \mathcal{F}_s]$
$+ \mathbb{E}[Z_k \cdot (M_t - M_{t_k}) | \mathcal{F}_s] = 0.$

Since • $\mathbb{E}[Z_i (M_{t_{i+1}} - M_{t_i}) | \mathcal{F}_s]$
$= \mathbb{E}[Z_i \cdot \mathbb{E}[M_{t_{i+1}} - M_{t_i} | \mathcal{F}_{t_i}] | \mathcal{F}_s]$
$= 0 \quad \forall\, j+1 \leq i \leq k-1$.

• $\mathbb{E}[Z_j (M_{t_{j+1}} - M_s) | \mathcal{F}_s] = Z_j \cdot \mathbb{E}[M_{t_{j+1}} - M_s | \mathcal{F}_s] = 0$
• $\mathbb{E}[Z_k \cdot (M_t - M_{t_k}) | \mathcal{F}_s] = \mathbb{E}[Z_k \cdot \mathbb{E}[M_t - M_{t_k} | \mathcal{F}_{t_k}] | \mathcal{F}_s]$
$= 0.$

$\Rightarrow H \cdot M$ is a MG.

Step 2: $H \cdot M$ is L^2 bounded.
If $j < k$, then we have that
$\mathbb{E}[Z_j (M_{t_{j+1}} - M_{t_j}) Z_k \cdot (M_{t_{k+1}} - M_{t_k})]$
$= \mathbb{E}[Z_j (M_{t_{j+1}} - M_{t_j}) \mathbb{E}[Z_k \cdot (M_{t_{k+1}} - M_{t_k}) | \mathcal{F}_{t_k}]]$
$= 0.$
So, $\mathbb{E}[(H \cdot M)_t^2] = \mathbb{E}[(\sum_{k=0}^{n-1} Z_k \cdot (M_{t_{k+1} \wedge t} - M_{t_k \wedge t}))^2]$
$= \mathbb{E}[\sum_{k=0}^{n-1} Z_k^2 \cdot (M_{t_{k+1} \wedge t} - M_{t_k \wedge t})^2]$
$\leq \|H\|_\infty^2 \cdot \sum_{k=0}^{n-1} \mathbb{E}[(M_{t_{k+1} \wedge t} - M_{t_k \wedge t})^2]$
$\leq 4 \cdot \|H\|_\infty^2 \cdot \mathbb{E}[(M_\infty - M_0)^2]$ (Doob's L^2-ineq.)
MG orthogonality again to telescope.
This bound is uniform in t, so $H \cdot M$ is L^2 bounded, $H \cdot M \in \mathcal{M}^2$.

Step 3:
$$\mathbb{E}[(H \cdot M)_\infty^2] \leq \liminf_{t \to \infty} \mathbb{E}[(H \cdot M)_t^2]$$
$$\leq \sup_{t \geq 0} \mathbb{E}[(H \cdot M)_t^2] \leq 4 \cdot \|H\|_\infty^2 \cdot \mathbb{E}[(M_\infty - M_0)^2]$$
□

Space of integrators: For X cadlag and adapted, define the norm:
$|||X||| = \|X^*\|_{L^2}$, $X^* = \sup_{t \geq 0} |X_t|$.
$\mathcal{C}^2 = \{\text{cadlag, adapted processes } X \text{ with } |||X||| < \infty\}$

Define the norm on $\mathcal{M}^2$ is given by
$\|X\| = \|X_\infty\|_{L^2}$.
Clearly a seminorm. To see that it is a norm, suppose that $\|X\| = \|X_\infty\|_{L^2} = 0$
$\Rightarrow X_\infty = 0$ a.s.
$\Rightarrow X_t = \mathbb{E}[X_\infty | \mathcal{F}_t] = 0$ a.s. $\forall t \geq 0$.
Cadlag property $\Rightarrow X \equiv 0$ a.s.

Set: $\mathcal{M} = \{\text{cadlag martingales}\}$
$\mathcal{M}_c = \{\text{cont. martingales}\}$
$\mathcal{M}_{c,loc} = \{\text{cont. loc. martingales}\}$.

LECTURE 7

X cadlag, adapted, $|||X||| = ||X^*||_{L^2}$,
$X^* = \sup_{t\geq 0} |X_t|$

$C^2 = \{$cadlag, adapted X with $|||X||| < +\infty\}$
$X \in \mathcal{M}^2$, $||X|| = ||X_\infty||_{L^2}$, $X_\infty = \lim_{t\to\infty} X_t$ "final value".
$\mathcal{M} = \{$cadlag MGs$\}$,
$\mathcal{M}_c = \{$Cont. MGs$\}$, $\mathcal{M}_{c,loc} = \{$cont. loc. MGs$\}$

<u>Proposition:</u>
a) $(C^2, |||\cdot|||)$ is complete.
b) $\mathcal{M}^2 = \mathcal{M} \cap C^2$
c) $(\mathcal{M}^2, ||\cdot||)$ is a Hilbert space,
$\mathcal{M}_c^2 = \mathcal{M}_c \cap \mathcal{M}^2$ is a closed subspace.
d) The map $\mathcal{M}^2 \to L^2(\mathcal{F}_\infty)$ is an isometry.
$X \mapsto X_\infty$
$\mathcal{F}_\infty = \sigma(\mathcal{F}_t : t > 0)$.

<u>Remark:</u> We can identify an element of $\mathcal{M}^2$ with its final value so $(\mathcal{M}^2, ||\cdot||)$ inherits the Hilbert space structure $(L^2(\mathcal{F}_\infty), ||\cdot||_{L^2})$. Since $(\mathcal{M}_c^2, ||\cdot||)$ is a closed, linear subspace of $(\mathcal{M}^2, ||\cdot||)$ by c), it is also a Hilbert space. This is the space of processes against which we will integrate.

<u>Proof:</u>
a) Suppose that (X^n) is Cauchy wrt $|||\cdot|||$. Then $\exists$ a subsequence $(X^{n_k})_{k=1}$ of (X^n) s.t. $\sum_k |||X^{n_k} - X^{n_{k+1}}||| < \infty$. Thus

$$\Big\|\sum_k \sup |X_t^{n_k} - X_t^{n_{k+1}}|\Big\|_{L^2} \leq \sum_k |||X^{n_k} - X^{n_{k+1}}||| < \infty$$

$$\Rightarrow \sum_{k\geq 0} \sup_{t\geq 0} |X_t^{n_k} - X_t^{n_{k+1}}| < \infty \text{ a.s.}$$

$\Rightarrow (X^{n_k})_{k\geq 1}$ is uniformly Cauchy on $[0,\infty)$ a.s., hence converges to a cadlag limit X.

<u>NTS:</u> $X^n \to X$ wrt $|||\cdot|||$.

$$|||X - X^n|||^2 = \mathbb{E}\Big[\sup_{t\geq 0} |X_t - X_t^n|^2\Big]$$

$$= \mathbb{E}\Big[\lim_{k\to\infty} \sup_{t\geq 0} |X_t^n - X_t^{n_k}|^2\Big]$$

Fatou

$$\leq \liminf_{k\to\infty} \mathbb{E}\Big[\sup_{t\geq 0} |X_t^n - X_t^{n_k}|^2\Big]$$

$$\leq \Big(\liminf_{k\to\infty} |||X^n - X^{n_k}|||^2\Big) \xrightarrow{n\to\infty} 0 \text{ a.s.}$$

Since X^n is Cauchy.

b) Suppose that $X \in C^2 \cap \mathcal{M}$. Then $|||X||| < +\infty$. So, $\sup_{t\geq 0} ||X_t||_{L^2} \underset{\text{Jensen}}{\leq} ||\sup_{t\geq 0}|X_t|||_{L^2}$
$= |||X||| < +\infty \Rightarrow X \in \mathcal{M}^2$.

Suppose that $X \in \mathcal{M}^2$. By Doob's L^2-inequality, $|||X||| \leq 2 \cdot ||X_\infty||_{L^2} = 2||X|| < \infty \Rightarrow X \in C^2 \cap \mathcal{M}$.
$\Rightarrow \mathcal{M}^2 = \mathcal{M} \cap C^2$.

c) Note that $(X,Y) \mapsto \mathbb{E}[X_\infty \cdot Y_\infty]$ defines an inner product on $\mathcal{M}^2$. For $X \in \mathcal{M}^2$,
$||X|| \leq |||X||| \leq 2 \cdot ||X||$
$\uparrow$ Doob's L^2-ineq.
$\Longrightarrow ||\cdot||, |||\cdot|||$ are equivalent norms on $\mathcal{M}^2$.
To show that $(\mathcal{M}^2, ||\cdot||)$ is complete, it suffices to show that $(\mathcal{M}^2, |||\cdot|||)$ is complete. To see this, let X^n be a sequence in $\mathcal{M}^2$ s.t. $|||X^n - X||| \to 0$ as $n \to +\infty$, where $X \in C^2$. (Suffices to show $\mathcal{M}^2$ is closed).
We know that X cadlag, adapted, L^2-bounded since $X \in C^2$.

<u>NTS:</u> $X \in \mathcal{M}^2$. Fix $s<t$, we have that:

$$||\mathbb{E}[X_t | \mathcal{F}_s] - X_s||_{L^2} \underset{X^n \text{ MG}}{=} ||\mathbb{E}[X_t - X_t^n | \mathcal{F}_s] + X_s^n - X_s||_{L^2}$$

$$\overset{\Delta}{\leq} ||\mathbb{E}[X_t - X_t^n | \mathcal{F}_s]||_{L^2} + ||X_s^n - X_s||_{L^2}$$

Jensen

$$\leq ||X_t^n - X_t||_{L^2} + ||X_s^n - X_s||_{L^2} \leq 2 \cdot |||X^n - X||| \xrightarrow{n\to\infty} 0$$

$\Rightarrow X \in \mathcal{M}^2 \Rightarrow \mathcal{M}^2$ is closed in C^2

d) True by definition $\square$.

<u>Space of integrals:</u>
Let (X^n) be a sequence of processes. We say that $X^n \to X$ uniformly on compact sets in probability (ucp) if for every $\varepsilon > 0$,
$\mathbb{P}\big(\sup_{s\leq t} |X_s^n - X_s| > \varepsilon\big) \to 0$ as $n \to +\infty$.

<u>Theorem:</u> Suppose that $M \in \mathcal{M}_{c,loc}$. There exists a unique (up to indistinguishability), continuous, adapted non-decreasing process $[M_t]$ s.t. $[M_0] = 0$, $M^2 - [M] \in \mathcal{M}_{c,loc}$.
Moreover, if we set:

$$[M]_t^n = \sum_{k=0}^{\lceil 2^n t\rceil - 1} \big(M_{(k+1)\cdot 2^{-n}} - M_{k\cdot 2^{-n}}\big)^2,$$

then $[M]^n \to [M]$ ucp as $n \to \infty$. The process $[M]$ is called the <u>quadratic variation</u> of M.

<u>Example:</u> Let B be a standard BM. Then $(B_t^2 - t)_{t\geq 0}$ is a MG $\Rightarrow [B_t] = t$. We will prove later that Brownian Motion is characterised by this property, i.e.
$M \in \mathcal{M}_{c,loc}$, $M_0 = 0$, $[M]_t = t$ $\forall t \geq 0$, then M is a BM. (Lévy characterisation of BM).

<u>Proof:</u> Replace M_t with $M_t - M_0$ so WLOG $M_0 = 0$
<u>Step 1:</u> Uniqueness. Suppose that A, A' are two non-decreasing, continuous, adapted processes satisfying the conditions in the thm.
Then, $A_t - A_t' = (M_t^2 - A_t') - (M_t^2 - A_t)$
<u>LHS:</u> cont., bdd variation
<u>RHS:</u> process in $\mathcal{M}_{c,loc}$
$\Rightarrow A - A'$ constant, $A_0 = A_0' = 0 \Rightarrow A = A'$.

(continuing next time)

LECTURE 8

<u>Theorem</u>: Let $M \in \mathcal{M}_{c,loc}$. There exists (up to indistinguishability) a unique adapted, continuous, non-decreasing process $[M]$ s.t. $[M]_0 = 0$ and $M^2 - [M] \in \mathcal{M}_{c,loc}$.
Moreover, if we set
$$[M]^n_t = \sum_{k=0}^{\lceil 2^n t \rceil - 1} (M_{(k+1)2^{-n}} - M_{k \cdot 2^{-n}})^2$$
then $[M]^n \longrightarrow [M]$ ucp as $n \to \infty$.
The process $[M]$ is called the <u>quadratic variation</u> of M.

<u>Last time:</u> Uniqueness (Step 1). $\boxed{\text{Today}}$: existence.

<u>Lemma</u>: Suppose that $M \in \mathcal{M}$ is bounded. Then for any $N \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots < t_N < \infty$, we have that:
$$\mathbb{E}\Big[\Big(\sum_{k=0}^{N-1} \underbrace{(M_{t_{k+1}} - M_{t_k})}_{\Delta_k}{}^2\Big)^2\Big] \le 48 \times \|M\|^4_{L^\infty}$$

<u>Proof</u>: $\mathbb{E}\big[\big(\sum_{k=0}^{N-1} (\Delta_k)^2\big)^2\big]$
$$\circledast = \sum_{k=0}^{N-1} \mathbb{E}[(\Delta_k)^4] + 2\sum_{k=0}^{N-1} \mathbb{E}\Big[\Delta_k^2 \sum_{j=k+1}^{N-1} \Delta_j^2\Big]$$

For each fixed k, we have that:
$$\mathbb{E}\Big[\Delta_k^2 \sum_{j=k+1}^{N-1} \Delta_j^2\Big]$$
$$= \mathbb{E}\Big[\Delta_k^2\, \mathbb{E}\Big[\sum_{j=k+1}^{N-1} \Delta_j^2 \,\Big|\, \mathcal{F}_{t_{k+1}}\Big]\Big]$$
$$= \mathbb{E}\Big[\Delta_k^2 \cdot \mathbb{E}\Big[\Big(\sum_{j=k+1}^{N-1} \Delta_j\Big)^2 \,\Big|\, \mathcal{F}_{t_{k+1}}\Big]\Big]$$
(MG orthogonality)
$$= \mathbb{E}\big[\Delta_k^2 \cdot \mathbb{E}[(M_{t_N} - M_{t_{k+1}})^2 \mid \mathcal{F}_{t_{k+1}}]\big]$$
$$= \mathbb{E}\big[\Delta_k^2 \cdot (M_{t_N} - M_{t_{k+1}})^2\big]$$

$$\Rightarrow \circledast \le \mathbb{E}\Big[\Big(\max_{0 \le j \le N-1} |M_{t_{j+1}} - M_{t_j}|^2 + 2 \cdot \max_{0 \le j \le N-1} |M_{t_N} - M_{t_j}|^2\Big) \times \Big(\sum_{k=0}^{N-1} \Delta_k^2\Big)\Big]$$
$$\le 12 \cdot \|M\|^2_{L^\infty} \cdot \mathbb{E}\Big[\sum_{k=0}^{N-1} \Delta_k^2\Big]. \quad (a+b)^2 \le 2(a^2+b^2).$$
$$= 12 \cdot \|M\|^2_{L^\infty} \cdot \mathbb{E}\Big[\Big(\sum_{k=0}^{N-1} \Delta_k\Big)^2\Big]$$
$$= 12 \cdot \|M\|^2_{L^\infty} \cdot \mathbb{E}[(M_{t_N} - M_{t_0})^2]$$
$$\le 48 \cdot \|M\|^4_{L^\infty} \quad \square$$

<u>Proof of the existence of QV:</u>
WLOG $M_0 = 0$ (by replacing M_t with $M_t - M_0$ if necessary).
<u>Step 2:</u> $M \in \mathcal{M}_c$ bounded ($M \in \mathcal{M}_c^2$).
Fix $T > 0$ and set:
$$H^n_t = \sum_{k=0}^{\lceil 2^n T \rceil - 1} M_{k 2^{-n}} \mathbb{1}_{(k 2^{-n}, (k+1) 2^{-n}]}(t).$$
Then $H^n \in \mathcal{S}$ for all n and set:
$$X^n_t = (H^n \cdot M)_t = \sum_{k=0}^{\lceil 2^n T \rceil - 1} M_{k \cdot 2^{-n}} (M_{(k+1)2^{-n} \wedge t} - M_{k \cdot 2^{-n} \wedge t})$$

Then $X^n \in \mathcal{M}_c$, bounded $\Rightarrow X^n \in \mathcal{M}_c^2$. Will show that (X^n) is Cauchy in $(\mathcal{M}_c^2, \|\cdot\|)$ hence has a limit in $\mathcal{M}_c^2$. Fix $n \ge m \ge 1$ and write $H = H^n - H^m$ so that: $X^n - X^m = (H^n - H^m) \cdot M = H \cdot M$.
Then,
$$\|X^n - X^m\|^2 = \mathbb{E}[(H \cdot M)^2_\infty] = \mathbb{E}[(H \cdot M)^2_T]$$
$$= \mathbb{E}\Big[\Big(\sum_{k=0}^{\lceil 2^n T \rceil - 1} H_{k \cdot 2^{-n}} (M_{(k+1) \cdot 2^{-n}} - M_{k \cdot 2^{-n}})\Big)^2\Big]$$
$$= \mathbb{E}\Big[\sum_{k=0}^{\lceil 2^n T \rceil - 1} H^2_{k \cdot 2^{-n}} (M_{(k+1) \cdot 2^{-n}} - M_{k \cdot 2^{-n}})^2\Big]$$
(MG orthogonality).
$$\le \mathbb{E}\Big[\sup_{t \in [0,T]} (H_t)^2 \cdot \sum_{k=0}^{\lceil 2^n T \rceil - 1} (M_{(k+1) \cdot 2^{-n}} - M_{k \cdot 2^{-n}})^2\Big]$$
$$\le \Big(\mathbb{E}\Big[\sup_{t \in [0,T]} |H_t|^4\Big]\Big)^{1/2} \times \mathbb{E}\Big[\Big(\sum_{k=0}^{\lceil 2^n T \rceil - 1} (M_{(k+1) 2^{-n}} - M_{k \cdot 2^{-n}})^2\Big)^2\Big]^{1/2}$$

<u>First term:</u> (A) $\sup_{t \in [0,T]} |H_t|^4 = \sup_{t \in [0,T]} |H^n_t - H^m_t|^4 \le 16 \cdot \|M\|^4_{L^\infty}$
(B) $\sup_{t \in [0,T]} |H^n_t - H^m_t| \to 0$ as $n, m \to \infty$

since M is continuous. By the Bounded Convergence Theorem, first term $\to 0$ as $n, m \to \infty$.

Second term $\le (48 \cdot \|M\|^4_{L^\infty})^{1/2} < \infty$
$\Rightarrow \|X^n - X^m\| \to 0$ as $n, m \to \infty$. Since $(\mathcal{M}_c^2, \|\cdot\|)$ is complete, $\exists Y \in \mathcal{M}_c^2$ s.t. $X_n \to Y$ as $n \to \infty$ in $\mathcal{M}_c^2$.

For any n and $1 \le k \le \lceil 2^n T \rceil$, we have that
$$M^2_{k \cdot 2^{-n}} - 2 \cdot X^n_{k \cdot 2^{-n}} = \sum_{j=0}^{k-1} (M_{(j+1) 2^{-n}} - M_{j \cdot 2^{-n}})^2 = [M]^n_{k \cdot 2^{-n}}$$
$\Rightarrow \forall n$, $M^2 - 2X^n$ is non-decreasing when restricted to times of the form $\{k \cdot 2^{-n} : 1 \le k \le \lceil 2^n T \rceil\}$.
To prove the same is also true for $M^2 - 2Y$ it suffices to show that $X^n \to Y$ a.s. uniformly, at least along a subsequence.
This follows from the equivalence of norms $\|\cdot\|$, $|||\cdot|||$.

Set $[M]_t = M_t^2 - 2Y_t$. Then $[M]$ is continuous, adapted, non-decreasing and $M^2 - [M] = 2Y \in \mathcal{M}_c^2$.

Can extend to all times by applying the above $T = k$ $\forall k \in \mathbb{N}$. Uniqueness $\Rightarrow$ process obtained with $T = k$, $T = k+1$ restricted to $[0, k]$ is the same.

<u>Step 3:</u> $[M]^n \to [M]$ ucp as $n \to \infty$.
$X^n \to Y$ in $(\mathcal{M}_c^2, \|\cdot\|) \Rightarrow \sup_{0 \le t \le T} |X^n_t - Y_t| \to 0$ as $n \to \infty$ in L^2 since $\|\cdot\|$, $|||\cdot|||$ are equivalent $\Rightarrow \sup_{0 \le t \le T} |X^n_t - Y_t| \to 0$ in probability.
Now, $[M]^n_t = M^2_{2^{-n} \lceil 2^n t \rceil} - 2X^n_{2^{-n} \lceil 2^n t \rceil}$
So, $\sup_{0 \le t \le T} |[M]_t - [M]^n_t|$
$$\le \sup_{0 \le t \le T} |M^2_{2^{-n} \lceil 2^n t \rceil} - M_t^2|$$
$$+ 2 \cdot \sup_{0 \le t \le T} |X^n_{2^{-n} \lceil 2^n t \rceil} - Y_{2^{-n} \lceil 2^n t \rceil}|$$
$$+ 2 \cdot \sup_{0 \le t \le T} |Y_{2^{-n} \lceil 2^n t \rceil} - Y_t|$$
Each term on RHS $\to 0$ in probability. ($\Rightarrow$ convergence in ucp).

LECTURE 9

Step 4: $M \in \mathcal{M}_{c,loc}$. ["Localisation argument"]
For each $n \geq 1$, let $T_n = \inf\{t \geq 0 : |M_t| \geq n\}$. Then (T_n) reduces M and M^{T_n} is bounded MG for all n. Therefore $\exists!$ continuous, adapted and non-decreasing process $[M^{T_n}]$ such that $[M^{T_n}]_0 = 0$ and, $(M^{T_n})^2 - [M^{T_n}] \in \mathcal{M}_{c,loc}$.
Let $A^n = [M^{T_n}]$. By uniqueness, $(A^{n+1}_{t\wedge T_n})$, (A^n_t) are indistinguishable. Let A be the process such that
$A_{t\wedge T_n} = A^n_t \quad \forall n$. Then $M^2_{t\wedge T_n} - A_{t\wedge T_n} \in \mathcal{M}_c$ for all $n \in \mathbb{N}$. $\Rightarrow M^2 - A \in \mathcal{M}_{c,loc}$ with reducing sequence (T_n). $\Rightarrow [M] = A$.

Know: $[M^{T_k}]^n \to [M^{T_k}]$ UCP as $n \to \infty$ $\forall k$, via. $\forall \varepsilon, T > 0$: $\mathbb{P}\left[\sup_{0\leq t\leq T} \left|[M^{T_k}]^n_t - [M^{T_k}]_t\right| > \varepsilon\right] \to 0$

as $n \to \infty$. On $\{T_k > T\}$, $[M]^n_t = [M^{T_k}]^n_t$ and $[M]_t = [M^{T_k}]_t$.
Thus: $\mathbb{P}\left[\sup_{0\leq t\leq T} |[M]^n_t - [M]_t| > \varepsilon\right]$

$\leq \mathbb{P}[T_k \leq T] + \mathbb{P}\left[\sup_{0\leq t\leq T} |[M^{T_k}]^n_t - [M^{T_k}]_t| > \varepsilon\right]$

$\to 0$ as $n \to \infty$, then $k \to \infty$.
LHS $\to 0$ as $n \to \infty$. $\square$

Theorem: Let $M \in \mathcal{M}^2_c$. Then $M^2 - [M]$ is a UI MG.
Proof: let $T = \inf\{t \geq 0 : [M]_t \geq n\}$ for $n \in \mathbb{N}$. Then $T_n \uparrow \infty$ as $n \to \infty$, T_n is a stopping time, $[M]_{t\wedge T_n} \leq n$

$$\Big|\underbrace{M^2_{t\wedge T_n} - [M]_{t\wedge T_n}}_{\in\, \mathcal{M}_{c,loc}}\Big| \leq n + \sup_{u\geq 0} M_u^2$$

Doob's inequality $\Rightarrow$ RHS is integrable
$\Rightarrow M^2_{t\wedge T_n} - [M]_{t\wedge T_n} \in \mathcal{M}_c$.

OST $\Rightarrow \mathbb{E}[M^2_{t\wedge T_n} - [M]_{t\wedge T_n}] = 0$
$\Rightarrow \mathbb{E}[[M]_{t\wedge T_n}] = \mathbb{E}[M^2_{t\wedge T_n}]$
Send $t \to \infty$. Monotone convergence thm
$\Rightarrow$ LHS $\to \mathbb{E}[[M]_{T_n}]$
RHS $\Rightarrow$ Dominated Convergence theorem
$\Rightarrow$ RHS $\to \mathbb{E}[M^2_{T_n}]$
$\Rightarrow \mathbb{E}[[M]_{T_n}] = \mathbb{E}[M^2_{T_n}]$
Send $n \to \infty$. MCT $\Rightarrow$ LHS $\to \mathbb{E}[[M]_\infty]$
RHS $\Rightarrow$ RHS $\to \mathbb{E}[M^2_\infty]$
$\Rightarrow \mathbb{E}[[M]_\infty] = \mathbb{E}[M^2_\infty] < \infty$
$\Rightarrow \mathbb{E}[[M]_\infty]$ is integrable.

Moreover, $|M_t^2 - [M]_t| \leq \sup_{u\geq 0} M_u^2 + [M]_\infty$
RHS is integrable $\Rightarrow M_t^2 - [M]_t \in \mathcal{M}_c$ and UI as it is dominated by an integrable r.v. $\square$

The space $L^2(M)$, $M \in \mathcal{M}^2_c$

Recall that $\mathcal{P}$ = previsible σ-algebra
$= \sigma(E \times (s,t] : E \in \mathcal{F}_s, s < t)$.
For $A \in \mathcal{P}$, define $\mu(A) = \mathbb{E}\left[\int_0^\infty \mathbb{1}_A(\omega, s)\, d[M]_s\right]$

Then μ is a measure on $(\Omega \times (0,\infty), \mathcal{P})$.
Moreover, it is uniquely determined by
$\mu(E \times (s,t]) = \mathbb{E}[\mathbb{1}_E([M]_t - [M]_s)]$
for $s < t$, $E \in \mathcal{F}_s$ since $\mathcal{P}$ is generated by sets of this form and they form a π-system. If $H \geq 0$ is previsible, then:
$$\int_{\Omega\times(0,\infty)} H\, d\mu = \mathbb{E}\left[\int_0^\infty H_s\, d[M]_s\right].$$

Definition: let $L^2(M) = L^2(\Omega \times (0,\infty), \mathcal{P}, \mu)$.
Write $\|H\|_{L^2(M)} = \|H\|_M = \left(\mathbb{E}\left[\int_0^\infty H_s^2\, d[M]_s\right]\right)^{1/2}$
$L^2(M)$ = previsible processes with $\|H\|_M < \infty$.
Hilbert space. This is the space of integrands

Remark: $(L^2(M), \|\cdot\|_M)$ depends on M since μ depends on M, but $\mathcal{S} \subseteq L^2(M)$ $\forall M \in \mathcal{M}^2_c$.
simple processes

Itô integrals: Recall that for
$H = \sum_{k=0}^{n-1} Z_k \cdot \mathbb{1}_{(t_k, t_{k+1}]} \in \mathcal{S}$, $M \in \mathcal{M}^2_c$,
we set $(H\cdot M)_t = \sum_{k=0}^{n-1} Z_k \cdot (M_{t_{k+1}\wedge t} - M_{t_k\wedge t}) \in \mathcal{M}^2_c$
This map defines a map:
$$\begin{array}{ccc} \mathcal{S} & \longrightarrow & \mathcal{M}^2_c \\ \cap & & \\ L^2(M) & & \end{array}$$
Will prove that it defines an isometry between $(L^2(M), \|\cdot\|_M)$ and $(\mathcal{M}^2_c, \|\cdot\|)$ when restricted to $\mathcal{S}$ (Itô isometry).

see calculation from before

$\|H\cdot M\|^2 = \|(H\cdot M)_\infty\|^2_{L^2}$
$= \sum_{k=0}^{n-1} \mathbb{E}[Z_k^2 (M_{t_{k+1}} - M_{t_k})^2]$
Since $M^2 - [M]$ is a MG, we have that:
$\mathbb{E}[Z_k^2 \cdot (M_{t_{k+1}} - M_{t_k})^2] = \mathbb{E}[Z_k^2 \cdot \mathbb{E}[(M_{t_{k+1}} - M_{t_k})^2 | \mathcal{F}_{t_k}]]$
$= \mathbb{E}[Z_k^2 \mathbb{E}[M_{t_{k+1}}^2 - M_{t_k}^2 | \mathcal{F}_{t_k}]]$
$= \mathbb{E}[Z_k^2\, \mathbb{E}[[M]_{t_{k+1}} - [M]_{t_k} | \mathcal{F}_{t_k}]]$
$= \mathbb{E}[Z_k^2 \cdot ([M]_{t_{k+1}} - [M]_{t_k})]$

$\Rightarrow \|H\cdot M\|^2 = \mathbb{E}\left[\sum_{k=0}^{n-1} Z_k^2([M]_{t_{k+1}} - [M]_{t_k})\right]$
$= \mathbb{E}\left[\int_0^\infty H_s^2\, d[M]_s\right] = \|H\|^2_M$.

LECTURE 10

Theorem (Ito isometry) There exist a unique isometry $I : L^2(M) \to \mathcal{M}_c^2$ such that $I(H) = H \cdot M$ for all $H \in S$.

Definition: for $M \in \mathcal{M}_c^2$, $H \in L^2(M)$, let $H \cdot M = I(H)$ where I is from the theorem. To prove the theorem, we first prove that the simple processes are dense in $L^2(M)$.

Lemma: let ν be any finite measure on $\mathcal{P}$. Then S is dense in $L^2(\mathcal{P}, \nu)$. In particular, if $M \in \mathcal{M}_c^2$ and we take $\nu = \mu$, we have that S is dense in $L^2(M)$.

Proof: Since $H \in S \Rightarrow \|H \cdot M\|_{L^\infty} < \infty$, it follows that $S \subseteq L^2(\mathcal{P}, \nu)$. Let $\bar{S}$ be the closure of S in $L^2(\mathcal{P}, \nu)$. WTS: $\bar{S} = L^2(\mathcal{P}, \nu)$.
Let $\mathcal{A} = \{A \in \mathcal{P} : \mathbb{1}_A \in \bar{S}\}$. WTS: $\mathcal{A} = \mathcal{P}$.
Obvious that $\mathcal{A} \in \mathcal{P}$. To see why the other direction holds, note that

(A) $\mathcal{A}$ contains the π-system $\{E \times (s,t] : E \in \mathcal{F}_s, s \le t\}$, which generates $\mathcal{P}$.

(B) $\mathcal{A}$ is a λ (d)-system

Dynkin's lemma $\Rightarrow \mathcal{P} \subseteq \mathcal{A} \Rightarrow \mathcal{A} = \mathcal{P}$.
$\Rightarrow$ Lemma follows since linear combinations of such indicators are dense in $L^2(\mathcal{P}, \nu)$. $\square$

Proof of Ito Isometry: Take $H \in L^2(M)$. Lemma $\Rightarrow \exists (H_n)$ in S s.t. $\|H^n - H\|_{L^2(M)} \to 0$, $n \to \infty$.
$\Rightarrow (H^n)$ is a Cauchy sequence wrt $\|\cdot\|_M$.
NTS: $I(H^n)$ is Cauchy wrt $\|\cdot\|$.

$$\|I(H^n) - I(H^m)\| = \|H^n \cdot M - H^m \cdot M\| \text{ (linearity)}$$
$$= \|(H^n - H^m) \cdot M\| = \|H^n - H^m\|_M \text{ (isometry)}.$$
$$\to 0 \text{ as } n, m \to \infty.$$

$\Rightarrow (I(H^n))$ converges wrt $\|\cdot\|$ to an element in $\mathcal{M}_c^2$ (since $(\mathcal{M}_c^2, \|\cdot\|)$ is complete). Set $I(H)$ to be this element.
NTS: I is well-defined.
Suppose that (K^n) in S converges to H wrt $\|\cdot\|_M$.

$$\|I(H^n) - I(K^n)\| = \|H^n \cdot M - K^n \cdot M\|$$
$$= \|H^n - K^n\|_M \le \|H^n - H\|_M + \|K^n - H\|_M \to 0$$

as $n \to \infty$ so that the limits of $I(H^n)$, $I(K^n)$ are indistinguishable.
NTS: I is an isometry $L^2(M) \to \mathcal{M}_c^2$
(H^n) in S, $H^n \to H \in L^2(M)$, $\|I(H)\| = \lim_{n \to \infty} \|H^n \cdot M\|$
$= \lim_{n \to \infty} \|H^n\|_M = \|H\|_M$. $\square$

$$I(H)_t = (H \cdot M)_t = \int_0^t H_s \, dM_s$$

This process $H \cdot M$ is the Itô (stochastic integral of H wrt M).

Extensions: Our goal now is to extend the definition of $H \cdot M$ to the setting that H is locally bounded and $M \in \mathcal{M}_{c,loc}$. Need to understand how the integral behaves under stopping.

Proposition: $H \in S$, $M \in \mathcal{M}$. Then for any stopping time T we have that $H \cdot (M^T) = (H \cdot M)^T$.
Proof: We have that:

$$(H \cdot (M^T))_t = \sum_{k=0}^{n-1} Z_k (M^T_{t_{k+1} \wedge t} - M^T_{t_k \wedge t})$$
$$= \sum_{k=0}^{n-1} Z_k (M_{t_{k+1} \wedge (T \wedge t)} - M_{t_k \wedge (T \wedge t)})$$
$$= (H \cdot M)_{t \wedge T} = (H \cdot M)^T_t. \quad \square$$

Proposition: let $M \in \mathcal{M}_c^2$, $H \in L^2(M)$, T stopping time. Then, $(H \cdot M)^T = (H \cdot \mathbb{1}_{(0,T]}) \cdot M = (H \cdot (M^T))$.
Proof: first note that if $H \in L^2(M)$, then: $H \cdot \mathbb{1}_{(0,T]} \in L^2(M)$ and $H \in L^2(M^T)$ so the integrals make sense.

Step 1: $H \in S$, $M \in \mathcal{M}_c^2$, T takes on finitely many values. Then $H \cdot \mathbb{1}_{(0,T]} \in S$ and $(H \cdot M)^T = (H \cdot \mathbb{1}_{(0,T]}) \cdot M = H \cdot M^T$.

Step 2: $H \in S$, $M \in \mathcal{M}_c^2$, T general stopping time.
Previous proposition $\Rightarrow (H \cdot M)^T = H \cdot (M^T)$.
NTS: $(H \cdot M)^T = (H \cdot \mathbb{1}_{(0,T]}) \cdot M$. Will prove via an approximation argument.
For $m, n \in \mathbb{N}$, let $T_{n,m} = (2^{-n} \lceil 2^n T \rceil) \wedge m$.
Then $T_{n,m}$ takes finitely many values and $T_{n,m} \searrow T \wedge m$ as $n \to \infty$. Thus

$$\|H \cdot \mathbb{1}_{(0, T_{n,m}]} - H \cdot \mathbb{1}_{(0, T \wedge m]}\|_M^2$$
$$= \mathbb{E}\left[\int_0^\infty H_t^2 \mathbb{1}_{(T \wedge m, T_{n,m}]} \, d[M]_t\right] \to 0$$

as $n \to \infty$ (DCT, dominating function: H_t^2).
$\Rightarrow (H \mathbb{1}_{(0, T_{n,m}]}) \cdot M \to (H \cdot \mathbb{1}_{(0, T \wedge m]}) \cdot M$ in $\mathcal{M}_c^2$ as $n \to \infty$.

Step 1: $LHS = (H \cdot M)^{T_{n,m}}$

$$(H \cdot M)^{T_{n,m}} \to (H \cdot M)^{T \wedge m}$$

pointwise a.s. by continuity of $H \cdot M$.

Thus, $(H \cdot \mathbb{1}_{(0, T \wedge m]}) \cdot M = (H \cdot M)^{T \wedge m}$.

LECTURE 11

Proposition: Let $M\in\mathcal{M}_c^2$, $H\in L^2(M)$, and let T be a stopping time. Then

$$(H\cdot M)^T = (H\cdot \mathbb{1}_{(0,T]})\cdot M = H\cdot M^T$$

Proof: **Step 1:** $H\in\mathcal{S}$, $H\in\mathcal{M}_c^2$, T takes finitely many values.

Step 2: $H\in\mathcal{S}$, $M\in\mathcal{M}_c^2$, T general stopping time.

Previous proposition $\Rightarrow (H\cdot M)^T = H\cdot M^T$. **NTS:** $(H\cdot M)^T = H\cdot\mathbb{1}_{(0,T]}\cdot M$

Let $n,m\in\mathbb{N}$ and $T_{n,m} = 2^{-n}\lceil 2^n T\rceil\wedge m$ so that $T_{n,m}$ takes on finitely many values. Moreover, $T_{n,m}\downarrow T\wedge m$ as $n\to\infty$.

Then: $\|H\cdot\mathbb{1}_{(0,T_{n,m}]} - H\cdot\mathbb{1}_{(0,T\wedge m]}\|_M^2$

$$= \mathbb{E}\left[\int_0^\infty H_t^2\cdot\mathbb{1}_{(T\wedge m,\,T_{n,m}]}\,d[M]_t\right]\to 0 \text{ as } n\to\infty$$

by the Dominated Convergence theorem with dominating function H_t^2.

Itô isometry $\Rightarrow H\mathbb{1}_{(0,T_{n,m}]}\cdot M\to H\cdot\mathbb{1}_{(0,T\wedge m]}\cdot M$ in $\mathcal{M}_c^2$ as $n\to\infty$.

By Step 1, $H\mathbb{1}_{(0,T_{n,m}]}\cdot M = (H\cdot M)^{T_{n,m}}\to (H\cdot M)^{T\wedge m}$ as $n\to\infty$ since $H\cdot M$ is continuous.

$\Rightarrow H\cdot\mathbb{1}_{(0,T\wedge m]}\cdot M = (H\cdot M)^{T\wedge m}\quad\forall m\geq 1.$

Repeat the same argument, send $m\to\infty$

$\Rightarrow H\cdot\mathbb{1}_{(0,T]}\cdot M = (H\cdot M)^T.$

Step 3: $H\in L^2(M)$, $M\in\mathcal{M}_c^2$, T general stopping time.

Let (H^n) be a sequence in $\mathcal{S}$ with $H^n\to H$ in $L^2(M)$.

Then: $\|(H^n\cdot M)^T - (H\cdot M)^T\| = \|(H^n\cdot M)_T - (H\cdot M)_T\|_{L^2}$ (final values)

$$\leq \Big\|\sup_{t\geq 0}|(H^n\cdot M)_t - (H\cdot M)_t|\Big\|_{L^2}$$

$$\leq 2\cdot\|(H^n\cdot M)_\infty - (H\cdot M)_\infty\|_{L^2}\quad(\text{Doob's } L^2\text{-ineq.})$$

$$= 2\cdot\|(H^n-H)\cdot M\| \underbrace{=}_{\text{Itô isometry}} 2\cdot\|H^n-H\|_M\to 0 \text{ as } n\to\infty$$

$\Rightarrow (H^n\cdot M)^T\longrightarrow (H\cdot M)^T$ in $\mathcal{M}_c^2$.

On the other hand,

$$\|H^n\cdot\mathbb{1}_{(0,T]} - H\cdot\mathbb{1}_{(0,T]}\|_M^2$$

$$= \mathbb{E}\left[\int_0^\infty (H_t^n - H_t)^2\cdot\mathbb{1}_{(0,T]}\,d[M]_t\right]$$

$$\leq \mathbb{E}\left[\int_0^\infty (H_t^n - H_t)^2\,d[M]_t\right] = \|H^n - H\|_M^2 \longrightarrow 0 \text{ as } n\to\infty.$$

$\Rightarrow H^n\cdot\mathbb{1}_{(0,T]}\cdot M\to H\cdot\mathbb{1}_{(0,T]}\cdot M$ in $\mathcal{M}_c^2$ by the Itô isometry. Since $(H^n\cdot M)^T = H^n\mathbb{1}_{(0,T]}\cdot M$ for all n, we have that $(H\cdot M)^T = H\mathbb{1}_{(0,T]}\cdot M$

NTS: $(H\cdot M)^T = (H\cdot M^T)$, assume $\exists (H^n)_n$ in $\mathcal{S}$ s.t. $H^n\longrightarrow H$ in $L^2(M)$.

$$\|H^n - H\|_{M^T}^2 = \mathbb{E}\left[\int_0^\infty (H_s^n - H_s)^2\,d[M^T]_s\right]$$

$$= \mathbb{E}\left[\int_0^\infty (H_s^n - H_s)^2\cdot\mathbb{1}_{(0,T]}\cdot d[M]_s\right]$$

$$\leq \|H^n - H\|_M^2\longrightarrow 0 \text{ as } n\to\infty.$$

$\Rightarrow H^n\cdot(M^T)\to H\cdot(M^T)$ in $\mathcal{M}_c^2$ by Itô isometry. Since $(H^n\cdot M)^T = H^n\cdot(M^T)$ for all n, we get that $(H\cdot M)^T = (H\cdot M^T)$ $\square$

Definition: We say that a previsible process H is **locally bounded** if $\exists$ a sequence $(S_n)_{n\in\mathbb{N}}$ of stopping times where $S_n\uparrow\infty$ as $n\to\infty$ and $H\cdot\mathbb{1}_{(0,S_n]}$ is bounded for all n.

Remark: Every continuous adapted process is previsible and locally bounded.

Definition: Let H be a locally bounded, previsible process with $H\mathbb{1}_{(0,S_n]}$ is bounded for all n where (S_n) is a sequence of stopping times with $S_n\uparrow\infty$ as $n\to\infty$. Let $M\in\mathcal{M}_{c,loc}$ with $M_0=0$ and let $S_n' = \inf\{t\geq 0: |M_t|\geq n\}$ so that $M^{S_n'}\in\mathcal{M}_c^2$ for all n. Let $T_n = S_n\wedge S_n'$ and set

$$(H\cdot M)_t = (H\cdot\mathbb{1}_{(0,T_n]}\cdot M^{T_n})_t\quad\forall t\in[0,T_n].$$

Using the previous proposition, this definition is well-defined and is consistent with the Itô integral with $M\in\mathcal{M}_c^2$, $H\in L^2(M)$.

Proposition: Let $M\in\mathcal{M}_{c,loc}$, H locally bounded and previsible, then $H\cdot M\in\mathcal{M}_{c,loc}$ where the sequence (T_n) is a reducing sequence. Moreover for any stopping time T, we have that

$$(H\cdot M)^T = H\cdot\mathbb{1}_{(0,T]}\cdot M = H\cdot M^T$$

Proof: That $H\cdot M\in\mathcal{M}_{c,loc}$ with reducing sequence (T_n) follows from the definition of $H\cdot M$. For any stopping time T,

$$(H\cdot M)^T = \lim_{n\to\infty}(H\cdot\mathbb{1}_{(0,T_n]}\cdot M^{T_n})^T$$

(pointwise limit)

(previous prop$^{\underline{n}}$) $= \lim_{n\to\infty}(H\cdot\mathbb{1}_{(0,T]}\cdot\mathbb{1}_{(0,T_n]}\cdot M^T)$

$$= H\cdot\mathbb{1}_{(0,T]}\cdot M$$

Same argument $\Rightarrow (H\cdot M)^T = H\cdot M^T$. $\square$

$H\cdot M\in\mathcal{M}_{c,loc}$

$[H\cdot M] = H^2\cdot[M]$.

Itô integral

Lebesgue-Stieltjes integral

LECTURE 12

Today: $[H\cdot M] = H^2\cdot[M]$, $H\cdot(K\cdot M) = (HK)\cdot M$, semimartingales.

Proposition: Let $M \in \mathcal{M}_{c,loc}$ and H locally bounded & previsible. Then $\underbrace{[H\cdot M]}_{\text{Itô}} = \underbrace{H^2\cdot[M]}_{\text{Lebesgue-Stieltjes}}$

Proof: Suppose that T is a bounded stopping time, H, M are uniformly bounded. Then:

$$\mathbb{E}[(H\cdot M)_T^2] = \mathbb{E}[(H\cdot 1\!\!1(0,T]\cdot M)^2_\infty]$$
$$(\text{Itô isometry}) = \mathbb{E}[(H^2\cdot 1\!\!1(0,T]\circ[M])_\infty]$$
$$= \mathbb{E}[(H^2\circ[M])_T]$$

OST: $(H\cdot M)^2 - H^2\cdot[M] \in \mathcal{M}_c$.

Uniqueness of quadratic variation $\Rightarrow$ $[H\cdot M] = H^2\cdot[M]$.

Now assume that H is locally bounded, previsible, $M \in \mathcal{M}_{c,loc}$. Let (T_n) be a sequence of stopping times so that $H 1\!\!1(0,T_n]$, M^{T_n} are bounded $\forall n$ and $T_n \uparrow \infty$ as $n\to\infty$.

$$[H\cdot M] = \lim_{n\to\infty} [H\cdot M]^{T_n}$$
$$= \lim_{n\to\infty} [(H\cdot M)^{T_n}] \quad \text{(uniqueness of quadratic variation)}$$
$$= \lim_{n\to\infty} [(H 1\!\!1(0,T_n]\cdot M^{T_n})]$$
$$= \lim_{n\to\infty} H^2 1\!\!1(0,T_n]\circ[M^{T_n}]$$
$$= H^2\circ[M] \quad \text{(applying MCT)} \qquad \square$$

Since $H\cdot M \in \mathcal{M}_{c,loc}$ for $M \in \mathcal{M}_{c,loc}$, H locally bounded, previsible, we can integrate against it.

Proposition: Let $M \in \mathcal{M}_{c,loc}$, H, K locally bounded, previsible. Then:

$$H\circ(K\circ M) = (HK)\circ M.$$

Proof: Elementary to check that this holds for H, K simple processes.(*) Now suppose that H, K, M are uniformly bounded.

NTS: $H \in L^2(K\cdot M)$, $HK \in L^2(M)$.

$$\|H\|^2_{L^2(K\cdot M)} = \mathbb{E}[(H^2\cdot[K\cdot M])_\infty]$$
$$= \mathbb{E}[(H^2\circ(K^2\circ[M]))_\infty]$$
$$(\text{Lebesgue-Stieltjes}) = \mathbb{E}[((HK)^2\circ[M])_\infty]$$
$$= \|HK\|^2_{L^2(M)}$$
$$\leq \min\{\|H\|^2_{L^\infty}\cdot\|K\|^2_{L^2(M)},\ \|K\|^2_{L^\infty}\cdot\|H\|^2_{L^2(M)}\}$$
$$< \infty$$

Let $(H^n), (K^n)$ be sequences in S which converge to H, K in $L^2(M)$ and wlog $(H^n), (K^n)$ uniformly bounded. Then:

$$H^n\circ(K^n\circ M) = (H^nK^n)\circ M$$

Then: $\|H^n\circ(K^n\circ M) - H\circ(K\circ M)\|$

$$\leq \|(H^n-H)\cdot(K^n\circ M)\| + \|H\circ((K^n-K)\circ M)\|$$
$$= \|H^n-H\|_{L^2(K^n M)} + \|H\cdot\|_{L^2((K^n-K)\cdot M)} \quad \text{(Itô isometry)}$$
$$(\text{see above}) = \|(H^n-H)\cdot K^n\|_{L^2(M)} + \|H\cdot(K^n-K)\|_{L^2(M)}$$
$$\leq \|K^n\|_{L^\infty}\cdot\|H^n-H\|_{L^2(M)} + \|H\|_\infty\cdot\|K^n-K\|_{L^2(M)}$$
$$\longrightarrow 0 \text{ as } n\to\infty.$$

Similar argument $\Rightarrow$ $(H^nK^n)\circ M \to (HK)\circ M$ as $n\to\infty$ in $\mathcal{M}_c^2$.

$\Rightarrow$ $H\circ(K\circ M) = (HK)\circ M$ (bounded case).

Now suppose that H, K are locally bounded, previsible and $M \in \mathcal{M}_{c,loc}$. Let (T_n) be a sequence of stopping times so that $H 1\!\!1(0,T_n]$, $K\cdot 1\!\!1(0,T_n]$, M^{T_n} are bounded and $T_n \uparrow \infty$ as $n\to\infty$.

Then $H\cdot 1\!\!1(0,T_n]\circ(K 1\!\!1(0,T_n]\circ M^{T_n})$
$$= HK 1\!\!1(0,T_n]\circ M^{T_n}$$

Also $K\cdot 1\!\!1(0,T_n]\circ M^{T_n} = (K\circ M)^{T_n}$. Hence,

$$H 1\!\!1(0,T_n]\circ(K 1\!\!1(0,T_n]\circ M^{T_n}) = H\cdot 1\!\!1(0,T_n]\circ(K\circ M)^{T_n}$$
$$= (H\circ(K\circ M))^{T_n} \to H\circ(K\circ M) \text{ as } n\to\infty.$$

Also, $(HK 1\!\!1(0,T_n])\circ M^{T_n} = (HK\circ M)^{T_n}$
$$\longrightarrow (HK\circ M) \text{ as } n\to\infty.$$

$$\Rightarrow H\circ(K\circ M) = (HK)\circ M \qquad \square$$

Remark: We have repeatedly used a "localisation" argument to reduce everything to the setting of a bounded integral and MG. This is a standard procedure; will omit in later arguments.

Semimartingales: **Definition:** A continuous, adapted process X is a semimartingale if it can be decomposed as

$$X = X_0 + M + A \quad , M \in \mathcal{M}_{c,loc}$$
$$A \text{ finite variation}$$
$$M_0 = A_0 = 0.$$

"Doob-Meyer decomposition".

For a continuous semiMG $X = X_0 + M + A$, define its quadratic variation by $[X] := [M]$.

Justified:

$$\sum_{k=0}^{\lceil 2^n t\rceil - 1} (X_{(k+1)\cdot 2^{-n}} - X_{k\cdot 2^{-n}})^2 \longrightarrow [M]_t$$

ucp as $n\to\infty$ (Ex. Sheet).

Definition: For H locally bounded and previsible, $X = X_0 + M + A$ cont. semiMG, define:

$$H\circ X = \underbrace{H\circ M}_{\text{Itô}} + \underbrace{H\circ A}_{\text{Leb.-Stieltjes}}$$

Then $H\circ X$ is a semiMG.

Proposition: Let X be a cont. semimartingale and H locally bounded, left-continuous & adapted. Then:

$$\sum_{k=0}^{\lceil 2^n t\rceil - 1} H_{k\cdot 2^{-n}}(X_{(k+1)2^{-n}} - X_{k\cdot 2^{-n}}) \longrightarrow (H\circ X)_t \quad \text{ucp as } n\to\infty.$$

Proof: See the typed lecture notes.

(*) By linearity in each argument, suffices to check for H, K consisting of single time intervals and noticing that for $0 \leq s'' < t''$, $0 \leq s' < t'$,

$$1\!\!1((s''\wedge t', t''\wedge t'] - (s''\wedge s', t''\wedge s'])$$
$$= 1\!\!1(s'', t'']\cdot 1\!\!1(s', t']$$

LECTURE 13

Summary of the Stochastic Integral

Step 1: $H \in \mathcal{S}$, $H_t = \sum_{k=0}^{n-1} Z_k \cdot \mathbb{1}_{(t_k, t_{k+1}]}(t)$,
Z_k bounded, $\mathcal{F}_{t_k}$-measurable, $M \in \mathcal{M}_c^2$ set:
$(H \cdot M)_t = \sum_{k=0}^{n-1} Z_k \cdot (M_{t_{k+1} \wedge t} - M_{t_k \wedge t})$.
Then $H \cdot M \in \mathcal{M}_c^2$.

Step 2: Equip $\mathcal{M}_c^2$ with a Hilbert space structure with norm $\|M\| = \|M_\infty\|_{L^2}$, $M \in \mathcal{M}_c^2$.

Step 3: Establish the existence of $[M]$, $M \in \mathcal{M}_{c,loc}$, where $[M]$ is the unique adapted, non-decreasing continuous process with $[M]_0 = 0$ so that $M^2 - [M] \in \mathcal{M}_{c,loc}$.

Step 4: For $M \in \mathcal{M}_c^2$, used $[M]$ to define a Hilbert space $(L^2(M), \|\cdot\|_M)$ where
$$\|H\|_M = \left(\mathbb{E}\left[\int_0^\infty H_s^2 \, d[M]_s\right]\right)^{1/2}$$

Step 5: Extend the integral to $H \in L^2(M)$, $M \in \mathcal{M}_c^2$ using the Ito isometry:
$\|H \cdot M\| = \|H\|_M$
$H \cdot M \in \mathcal{M}_c^2$ for all $H \in L^2(M)$, $M \in \mathcal{M}_c^2$.

Step 6: Extended to H locally bounded & previsible, $M \in \mathcal{M}_{c,loc}$ by setting $(H \cdot M)_t = (H \mathbb{1}_{(0,T_n]} \cdot M^{T_n})_t$ $\forall t \leq T_n$,

Step 7: Extend to H locally bounded, previsible and $X = X_0 + M + A$ a continuous semi MG by setting
$$H \cdot X = \underbrace{H \cdot M}_{\text{Ito}} + \underbrace{H \cdot A}_{\text{Lebesgue-Stieltjes}}$$
then $H \cdot X$ is a cont. semi MG.

Stochastic Calculus:

Definition: for $M, N \in \mathcal{M}_{c,loc}$ define the covariation of M, N by setting:
$$[M,N] = \frac{1}{4}([M+N] - [M-N])$$
(Polarization identity).
Note that: $[M,M] = [M]$.

Theorem: Let $M, N \in \mathcal{M}_{c,loc}$. Then:
a) $[M,N]$ is the unique process (up to indistinguishability) continuous, adapted, finite-variation process with $[M,N]_0 = 0$ so that $MN - [M,N] \in \mathcal{M}_{c,loc}$.
b) For $n \geq 1$, set
$$[M,N]_t^n = \sum_{k=0}^{\lceil 2^n t \rceil - 1} (M_{(k+1) \cdot 2^{-n}} - M_{k \cdot 2^{-n}}) \cdot (N_{(k+1) 2^{-n}} - N_{k \cdot 2^{-n}})$$
Then $[M,N]_t^n \longrightarrow [M,N]$ as $n \to \infty$ ucp.
c) If $M, N \in \mathcal{M}_c^2$, then $MN - [M,N]$ is a UI MG.
d) for H locally bounded, previsible
$$[H \cdot M, N] + [M, H \cdot N] = 2H \cdot [M,N]$$

Proof: a) $MN = 1/4((M+N)^2 - (M-N)^2)$ so
(*) $MN - [M,N] = 1/4(\underbrace{(M+N)^2 - [M+N]}_{\in \mathcal{M}_{c,loc}}) - 1/4(\underbrace{(M-N)^2 - [M-N]}_{\in \mathcal{M}_{c,loc}})$
$\Rightarrow MN - [M,N] \in \mathcal{M}_{c,loc}$. By definition, $[M,N]$ continuous, adapted and finite variation (difference of non-decreasing functions). Same argument used to prove the uniqueness of covariation.
b) Note: $[M,N]_t^n = 1/4([M+N]_t^n - [M-N]_t^n)$
$$\begin{array}{ccccc} & \downarrow \text{ucp} & & \downarrow \text{ucp} & \downarrow \text{ucp} \\ \Rightarrow & [M,N] & & [M+N] & [M-N] \end{array}$$
c) $MN - [M,N]$ is a UI MG for $M, N \in \mathcal{M}_c^2$ follows from (*) and the corresponding property for quadratic variation.
d) $[H \cdot (M +/- N)] = H^2 \cdot [M +/- N]$
$\Longrightarrow [H \cdot M, H \cdot N] = H^2 \cdot [M, N]$
Moreover, $(H+1)^2 \cdot [M,N] = [(H+1) \cdot M, (H+1) \cdot N]$
ES2 $= [H \cdot M + M, H \cdot N + N]$
$\xrightarrow{\text{bilinearity}} = [H \cdot M, H \cdot N] + [M, H \cdot N] + [H \cdot M, N] + \cancel{[M,N]}$
and $(H+1)^2 \cdot [M,N] = (H^2 + 2H + 1) \cdot [M,N]$
$= H^2 \cdot [M,N] + 2H \cdot [M,N] + \cancel{[M,N]}$.
$\Longrightarrow 2H \cdot [M,N] = [M, H \cdot N] + [H \cdot M, N]$ $\square$

Proposition: (Kunita-Watanabe identity)
Let $M, N \in \mathcal{M}_{c,loc}$, H locally bounded, previsible, then $[H \cdot M, N] = H \cdot [M,N]$.

Proof: NTS: $[H \cdot M, N] = [M, H \cdot N]$ as then we can apply part d) of the previous theorem.
Use that: $(H \cdot M)N - [H \cdot M, N] \in \mathcal{M}_{c,loc}$
$M(H \cdot N) - [M, H \cdot N] \in \mathcal{M}_{c,loc}$.
Will show that: $(H \cdot M)N - M(H \cdot N) \in \mathcal{M}_{c,loc}$
Suffices since then $[H \cdot M, N] - [M, H \cdot N] \in \mathcal{M}_{c,loc}$ with finite variation and starts from 0
$\Longrightarrow [H \cdot M, N] = [M, H \cdot N]$.
Localisation: WLOG $M, N \in \mathcal{M}_c^2$, H bounded.
By OST, suffices to show that for bounded stopping time T,
$$\mathbb{E}[(H \cdot M)_T N_T] = \mathbb{E}[M_T (H \cdot N)_T]$$
LHS $= \mathbb{E}[(H \cdot M)_\infty^T N_\infty^T]$, RHS $= \mathbb{E}[M_\infty^T (H \cdot N)_\infty^T]$
Suffices to show that $\mathbb{E}[(H \cdot M)_\infty N_\infty] = \mathbb{E}[M_\infty (H \cdot N)_\infty]$ (*)
$\forall M, N \in \mathcal{M}_c^2$, bounded H.
Suppose that $H = Z \cdot \mathbb{1}_{(s,t]}$, $Z \in \mathcal{F}_s$-measurable, bounded.
$\mathbb{E}[(H \cdot M)_\infty N_\infty] = \mathbb{E}[Z(M_t - M_s) N_\infty]$
$= \mathbb{E}[Z M_t \mathbb{E}[N_\infty | \mathcal{F}_t] - Z M_s \mathbb{E}[N_\infty | \mathcal{F}_s]]$
$= \mathbb{E}[Z(M_t N_t - M_s N_s)]$
same argument $\to \mathbb{E}[M_\infty (H \cdot N)_\infty] =$
Gives (*) for $H = Z \cdot \mathbb{1}_{(s,t]}$. Linearity gives (*) for $H \in \mathcal{S}$.

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Proof (Kunita-Watanabe): Last time: by localisation and OST, reduced the proposition to $\mathbb{E}[(H\cdot M_\infty)N_\infty] = \mathbb{E}[M_\infty (H\cdot N)_\infty]$ (*)
and proved (*) for $H = Z\cdot\mathbb{1}_{(s,t]}$ for $s<t$, Z $\mathcal{F}_s$-measurable, bounded. By linearity, (*) holds for all $H\in S$. Suppose that H is a bounded previsible process. Then there exists a sequence (H^n) in S so that $H^n \to H$ in $L^2(M)$, $L^2(N)$ [in the lemma where we showed that S are dense in $L^2(P,\nu)$, ν finite, to be given by $\nu(E) = \mathbb{E}\left[\int_0^\infty \mathbb{1}(E)\,(d[M]_s + d[N]_s)\right]$]
$\Rightarrow H^n\cdot M \to H\cdot M$, $H^n\cdot N \to H\cdot N$ in $\|\cdot\|$-norm.
$$\Rightarrow \begin{array}{l}(H^n\cdot M)_\infty \to (H\cdot M)_\infty \\ (H^n\cdot N)_\infty \to (H\cdot N)_\infty\end{array} \text{ and in } L^2 \text{ as } n\to\infty.$$
Thus,
$$\|((H^n\cdot M)_\infty - (H\cdot M)_\infty)N_\infty\|_{L^1} \overset{(C\text{-}S)}{\le} \|(H^n\cdot M)_\infty - (H\cdot M)_\infty\|_{L^2}\cdot\|N_\infty\|_{L^2} \longrightarrow \infty,\ n\to\infty.$$
$\Rightarrow \mathbb{E}[(H^n\cdot M)_\infty N_\infty] \to \mathbb{E}[(H\cdot M)_\infty N_\infty]$
Same works with M, N swapped $\Rightarrow$ (*) $\square$

Definition: For continuous semi-MGs X, Y define $[X,Y]$ to be the covariation of their MG parts.
- This is justified as:
$$[X,Y]_t^n \overset{\text{def}}{=} \sum_{k=0}^{\lceil 2^n t\rceil -1} (X_{(k+1)\cdot 2^{-n}} - X_{k\cdot 2^{-n}})(Y_{(k+1)\cdot 2^{-n}} - Y_{k\cdot 2^{-n}}) \longrightarrow [X,Y]_t \text{ ucp as } n\to\infty.$$
- Kunita-Watanabe also holds for semi MGs.

Proposition: Let X, Y be independent semi MGs, then their covariation $[X,Y]=0$.
Proof: ES2.

Itô's formula:

Theorem: (Integration by parts) Let X, Y be continuous semiMGs. Then:
$$(*)\quad X_tY_t - X_0Y_0 = \int_0^t X_s\,dY_s + \int_0^t Y_s\,dX_s + [X,Y]_t$$
Proof: Note that the integrals are well-defined since any continuous, adapted process is locally bounded and previsible.
Note that for $s\le t$, we have that
$$X_tY_t - X_sY_s = X_s(Y_t-Y_s) + Y_s(X_t-X_s) + (X_t-X_s)(Y_t-Y_s).$$
Since the LHS, RHS of (*) are continuous, suffices to prove the result for t of the form $t = m\cdot 2^{-j}$, $m, j\in\mathbb{N}$: $(n\ge j)$
$$X_tY_t - X_0Y_0 = \sum_{k=0}^{m\cdot 2^{n-j}-1}\Big(X_{k\cdot 2^{-n}}\cdot(Y_{(k+1)\cdot 2^{-n}} - Y_{k\cdot 2^{-n}}) + Y_{k\cdot 2^{-n}}\cdot(X_{(k+1)\cdot 2^{-n}} - X_{k\cdot 2^{-n}}) + (X_{(k+1)\cdot 2^{-n}} - X_{k\cdot 2^{-n}})(Y_{(k+1)\cdot 2^{-n}} - Y_{k\cdot 2^{-n}})\Big)$$
$$\longrightarrow (X\cdot Y)_t + (Y\cdot X)_t + [X,Y]_t \text{ ucp as } j\to\infty \quad\square$$
Note that $[X,Y]$ term does not appear if either X, Y are independent or if X or Y does not have a MG part.

Theorem: (Itô's formula) Let $(X^1,\dots,X^d)$ where each X^i for $1\le i\le d$ is a continuous semi MG. Let $f:\mathbb{R}^d\to\mathbb{R}$ be C^2. Then,
$$f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(X_s)\,dX_s^i + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i\partial x_j}(X_s)\,d[X^i,X^j]_s.$$

Remarks: (1) Integration by parts is a special case of Itô's formula with $f(x,y) = x\cdot y$.
(2) For $d=1$, Itô's formula is:
$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\,dX_s + \frac{1}{2}\int_0^t f''(X_s)\,d[X]_s.$$
Possible to derive using Taylor expansions since:
$$f(X_t) = f(X_0) + \sum_{k=0}^{\lfloor 2^n t\rfloor -1}\Big(f(X_{(k+1)\cdot 2^{-n}}) - f(X_{k\cdot 2^{-n}})\Big) + \Big(f(X_t) - f(X_{2^{-n}\lfloor 2^n t\rfloor})\Big)$$
$$= f(x_0) + \sum_{k=0}^{\lfloor 2^n t\rfloor -1} f'(X_{k\cdot 2^{-n}})(X_{(k+1)\cdot 2^{-n}} - X_{k\cdot 2^{-n}}) + \frac{1}{2}\sum_{k=0}^{\lfloor 2^n t\rfloor -1} f''(X_{k\cdot 2^{-n}})\cdot(X_{(k+1)\cdot 2^{-n}} - X_{k\cdot 2^{-n}})^2$$
$$+ \text{error} \longrightarrow f(X_0) + \int_0^t f'(X_s)\,dX_s + \frac{1}{2}\int_0^t f''(X_s)\,d[X]_s$$
ucp as $n\to\infty$.
Will prove it a different way since the extra error term is inconvenient to deal with.

Examples: (1) $X = B$, B standard Brownian motion, $f(x) = x^2$. Then:
$$f(X_t) = f(B_0) + \int_0^t f'(B_s)\,ds + \frac{1}{2}\int_0^t f''(B_s)\,d[B]_s$$
$$= 0 + \int_0^t 2B_s\,dB_s + \frac{1}{2}\int_0^t 2\,ds$$
$$= 2\int_0^t B_s\,dB + t$$
$\Rightarrow B_t^2 - t = 2\int_0^t B_s\,dB_s \in \mathcal{M}_{c,loc}$.

(2) Let $f:\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}$ be $C^{1,2}$ and $X_t = (t, B_t^1,\dots,B_t^d)$ where $B_t^1,\dots,B_t^d$ are independent Brownian motions. By Itô's formula,
$$f(t,\underline{B_t}) - f(0,\underline{B_0}) - \int_0^t \Big(\underbrace{\frac{\partial}{\partial t}}_{\text{first coord}} + \underbrace{\frac{\Delta}{2}}_{\text{last } d\text{-coords}}\Big) f(s,B_s)\,ds$$
$$= \sum_{i=1}^d \int_0^t \underbrace{\frac{\partial}{\partial x_i}}_{\text{in } i^{\text{th}} \text{ spatial coordinate}} f(s,B_s)\,dB_s^i \in \mathcal{M}_{c,loc}.$$
If f does not depend on t and is harmonic in spatial variables then $f(t,B_t)\in\mathcal{M}_{c,loc}$. If f is bounded, then $f(B_t)$ is a MG.

LECTURE 15

Proof (Itô's formula):
We are doing the proof for $d=1$; the case $d>1$ is left as an exercise. Let
$X = X_0 + M + A$ and let V be the total variation of A.
Let $T_r = \inf\{t \geq 0 : |X_t| + |V_t| + [M]_t > r\}$ for each $r>0$. Then (T_r) is a sequence of stopping times with $T_r \uparrow \infty$ as $r \to \infty$. It suffices to prove the formula in $[0, T_r]$ for each $r>0$. Let $\mathcal{A}$ be the subset of $C^2(\mathbb{R})$ so that the formula holds. WTS: $\mathcal{A} = C^2(\mathbb{R})$. Will prove by showing that:

a) $\mathcal{A}$ contains $f(x) \equiv 1$, $f(x) \equiv x$.
b) $\mathcal{A}$ is a vector space.
c) $\mathcal{A}$ is an algebra, i.e., $f, g \in \mathcal{A} \Rightarrow fg \in \mathcal{A}$.
d) If (f_n) is a sequence in $\mathcal{A}$ with $f_n \to f$ in $C^2(B_r)$ for each $r>0$ ($B_r = \{x \in \mathbb{R} : |x| < r\}$), then $f \in \mathcal{A}$.

Here, $B_r = \{|x|<r\}$ and $f_n \to f$ in $C^2(B_r)$ means that with

$$\Delta_{n,r} := \sup_{x\in B_r} |f_n - f| + \sup_{x\in B_r} |f_n' - f'| + \sup_{x \in B_r} |f_n'' - f''|$$

we have that $\Delta_{n,r} \to 0$ as $n \to \infty$ for each $r>0$.

a), b), c) $\Rightarrow$ polynomials are in $\mathcal{A}$. Wierestrass approximation theorem $\Rightarrow$ polynomials are dense in $C^2(B_r)$ $\forall r>0$, so d) $\Rightarrow \mathcal{A} = C^2(\mathbb{R})$.
That a), b) hold is easy to see.

Proof of c): suppose $f, g \in \mathcal{A}$. Let $F_t = f(X_t)$, $G_t = g(X_t)$. Itô's formula holds for $f, g \Longrightarrow F, G$ are continuous semi MG's.
Integration by parts:

$$F_t G_t - F_0 G_0 = \int_0^t F_s \, dG_s + \int_0^t G_s \, dF_s + \frac{1}{2}[F, G]_t$$

Since Itô's formula holds for g, we have that

$$\int_0^t F_s \, dG_s = \int_0^t F_s \, d\left(\int_0^s g'(X_u)\, dX_u + \int_0^s g''(X_s)\, d[X]_u\right)$$

$H \cdot (K \cdot M) = (HK) \cdot M$

$$(2) \quad = \int_0^t f(X_s) g'(X_s)\, dX_s + \frac{1}{2}\int_0^t f(X_s) g''(X_s)\, d[X]_u$$

Also,

$$(3) \quad \int_0^t G_s \, dF_s = \int_0^t f'(X_s) g(X_s)\, dX_s + \frac{1}{2}\int_0^t f''(X_s) g(X_s)\, d[X]_s.$$

$$\begin{aligned} [F, G]_t &= [f(X), g(X)]_t \\ &= [f'(X)\cdot X, g'(X) \cdot X] \leftarrow \text{by def}^n \text{ of cov. \& Itô formula.} \\ \text{(Kunita–Watanabe)} \quad &= \int_0^t f'(X_s) g'(X_s)\, d[X]_s. \quad (4) \end{aligned}$$

Plug (2)–(4) into (1) gives Itô's formula for fg, i.e. $fg \in \mathcal{A}$.

Proof of d): Suppose that (f_n) is a sequence in $\mathcal{A}$ and $f_n \to f$ in $C^2(B_r)$ $\forall r>0$.
WTS: Itô's formula for f, i.e. $f \in \mathcal{A}$.
Since Itô's formula holds for f_n:

$$\begin{aligned} f_n(X_t) &= f_n(X_0) + \int_0^t f_n'(X_s)\, dX_s + \frac{1}{2}\int_0^t f''(X_s)\, d[X]_s \\ &= f_n(X_0) + \left[\int_0^t f_n'(X_s)\, dA_s + \frac{1}{2}\int_0^t f''(X_s)\, d[X]_s\right] \\ &\quad + \int_0^t f'(X_s)\, dM_s. \end{aligned}$$

For the finite variation part:

$$\begin{aligned} &\int_0^{t\wedge T_r} |f_n'(X_s) - f'(X_s)|\, dV_s + \frac{1}{2}\int_0^{t \wedge T_r} |f_n''(X_s) - f_n''(X_s)|\, d[M]_s \\ &\leq \Delta_{n,r} \cdot \left(V_{t\wedge T_r} + \frac{1}{2}[M]_{t \wedge T_r}\right) \\ &\leq 2r\Delta_{n,r} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

$$\Longrightarrow \int_0^{t\wedge T_r} f_n'(X_s)\, dA_s + \frac{1}{2}\int_0^{t\wedge T_r} f_n''(X_s)\, d[M]_s$$

$$\longrightarrow \int_0^{t\wedge T_r} f'(X_s)\, dA_s + \frac{1}{2}\int_0^{t\wedge T_r} f''(X_s)\, d[M]_s$$

uniform in t.

MG part: $M^{T_r} \in \mathcal{M}_c^2$ since $[M^{T_r}] \leq r$.

$$\begin{aligned} &\|(f_n'(X)\cdot M)^{T_r} - (f'(X)\cdot M)^{T_r}\|^2 \\ &= \mathbb{E}\left[\int_0^{T_r} (f_n'(X_s) - f'(X_s))^2\, d[M]_s\right] \\ &\leq \Delta_{n,r}^2 \cdot \mathbb{E}\left[[M]_{T_r}\right] \leq r\Delta_{n,r}^2 \to 0, \; n \to \infty. \end{aligned}$$

$\Longrightarrow (f_n'(X)\cdot M)^{T_r} \to (f'(X)\cdot M)^{T_r}$ in $\mathcal{M}_c^2$ as $n \to \infty$.

$$\Longrightarrow f(X_{t \wedge T_r}) = f(X_0) + \int_0^{t\wedge T_r} f'(X_s)\, dX_s + \int_0^{t \wedge T_r} f''(X_s)\, d[X]_s$$

$\square$

Stratonovich Integral:
Let X, Y be continuous semi MGs. The Stratonovich Integral of X against Y is defined as:

$$\int_0^t X_s \, \partial Y_s = \underbrace{\int_0^t X_s \, dY_s}_{\text{Itô}} + \frac{1}{2}[X, Y]_t$$

"midpoint approximation"

$$\sum_{k=0}^{\lfloor 2^n t\rfloor - 1} \frac{(X_{(k+1)\cdot 2^{-n}} + X_{k\cdot 2^{-n}})}{2}(Y_{(k+1)\cdot 2^{-n}} - Y_{k \cdot 2^{-n}}).$$

$$\xrightarrow[\text{ucp}]{} \int_0^t X_s \, \partial Y_s.$$

Proposition: Let $X^1, \dots, X^d$ be cont. semi MGs and let $f: \mathbb{R}^d \to \mathbb{R}$ be C^3. Then

$$f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(X_s)\, \partial X_s^i.$$

In particular, integration by parts is given by:

$$X_t Y_t - X_0 Y_0 = \int_0^t X_s \, \partial Y_s + \int_0^t Y_s \, \partial X_s$$

$\Longrightarrow$ Stratonovich satisfies the usual rules of calculus. But the Stratonovich integral against $M \in \mathcal{M}_{c,loc}$ is <u>not</u> in $\mathcal{M}_{c,loc}$.
For example,

$$\int_0^t B_s \, \partial B_s = \underbrace{\int_0^t B_s \, dB_s}_{\text{Itô}} + \frac{1}{2}t$$

$$= \frac{1}{2}B_t^2 \notin \mathcal{M}_{c,loc}$$

for B a standard BM.

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Stratonovich Integral:

$$\int_0^t X_s \partial Y_s = \int_0^t X_s dY_s + \frac{1}{2}[X,Y]_t \text{ for } X, Y$$
cont. semi MGs.

Proposition: let $X^1, \dots, X^d$ be cont. semi MGs, $X = (X^1, \dots, X^d)$, and let $f: \mathbb{R}^d \to \mathbb{R}$ be C^3. Then
$$f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(X_s)\partial X_s^i$$

In particular, $X_t Y_t - X_0 Y_0 = \int_0^t X_s \partial Y_s + \int_0^t Y_s \partial X_s$

Proof: $d=1$; $d>1$ left as an exercise. Itô's formula:
$$(1)\ f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2}\int_0^t f''(X_s) d[X]_s$$
$$(2)\ f'(X_t) = f'(X_0) + \int_0^t f''(X_s) dX_s + \frac{1}{2}\int_0^t f'''(X_s) d[X]_s.$$
$$[f'(X), X] \underset{(2)}{=} [f''(X)\cdot X, X] \underset{\text{Kunita-Watanabe}}{=} f''(X)\cdot[X].$$
$$\to f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}[f'(X), X]$$
$$= f(X_0) + \int_0^t f'(X_s)\partial X_s \qquad \square$$

Shorthand:
$$Z_t = Z_0 + \int_0^t H_s dX_s \iff dZ_t = H_t dX_t$$
$$Z_t = Z_0 + \int_0^t H_s \partial X_s \iff \partial Z_t = H_t \partial X_t$$
$$Z_t = [X,Y]_t = \int_0^t d[X,Y]_s \iff dZ_t = dX_t dY_t$$

Computational rules:

$H_t d(K_t dX_t) = (H_t K_t) dX_t$ [iterated integral]
$H_t (dX_t dY_t) = d(H_t dX_t) dY_t$ [Kunita - Watanabe]
$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t$ [integration by parts]
$df(X_t) = \sum_{i=1}^d \frac{\partial}{\partial x_i} f(X_t) dX_t^i + \frac{1}{2}\sum_{i,j=1}^d \frac{\partial^2 f(X_t)}{\partial x_i \partial x_j} dX_t^i dX_t^j$
[Itô's formula].

Applications:

Theorem: (Lévy characterisation)
Let $X^1, \dots, X^d \in \mathcal{M}_{c,loc}$, $X = (X^1, \dots, X^d)$. Suppose $X_0 = 0$, $[X^i, X^j]_t = \delta_{ij} t \ \forall i,j, t \geq 0$. Then X is a standard BM.

Proof: NTS: for all $0 \leq s \leq t < \infty$, $X_t - X_s$ independent of $\mathcal{F}_s$ and has the law of $\mathcal{N}(0, (t-s)I_d)$, where I_d is the dxd identity matrix.
Equivalently,
$$\mathbb{E}[\exp(i\theta(X_t - X_s)) | \mathcal{F}_s] = \exp\left[-\frac{|\theta|^2}{2}(t-s)\right]$$
$\forall \theta \in \mathbb{R}^d$, $(\cdot,\cdot)$ = Euclidean inner product.
$|\theta|^2 = (\theta, \theta)$. (*)

For $\theta \in \mathbb{R}^d$, set $Y_t = (\theta, X_t) = \sum_{j=1}^d \theta^j X_t^j$.
Then $Y \in \mathcal{M}_{c,loc}$ since $\mathcal{M}_{c,loc}$ is a vector space.
Moreover, $[Y]_t = [Y,Y]_t = [\sum_{j=1}^d \theta^j X^j, \sum_{j=1}^d \theta^j X^j]_t$
$$= \sum_{j,k=1}^d \theta^j \theta^k [X^j, X^k]_t$$
$$= |\theta|^2 t$$
$$Z_t = \exp[iY_t + \tfrac{1}{2}[Y]_t] = \exp[i(\theta, X_t) + \tfrac{1}{2}|\theta|^2 t].$$

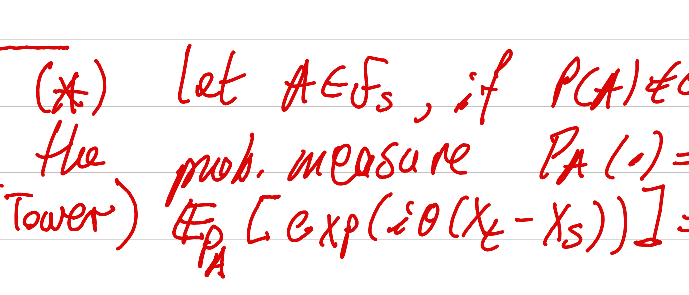
By Itô's formula applied to $W = iY_t + \frac{1}{2}[Y]$, $f(\omega) = e^\omega$, we have that:
$$dZ_t = Z_t(i\,dY_t + \tfrac{1}{2}d[Y]_t) - \tfrac{1}{2}Z_t d[Y]_t$$
$$= iZ_t dY_t$$
$\Rightarrow Z \in \mathcal{M}_{c,loc}$ since $Y \in \mathcal{M}_{c,loc}$.
Since Z is bounded on $[0,t]$ for $t \geq 0$, $Z \in \mathcal{M}_c$.
$\Rightarrow E[Z_t | \mathcal{F}_s] = Z_s$.
$\Rightarrow E[\exp[i\theta\cdot(X_t - X_s)] | \mathcal{F}_s] = \exp(-\frac{1}{2}|\theta|^2(t-s))$ $\square$

Theorem: (Dubins-Schwarz) Let $M \in \mathcal{M}_{c,loc}$ with $M_0 = 0$, $[M]_\infty = \infty$. Set $\tau_s = \inf\{t \geq 0 : [M]_t > s\}$, $B_s = M_{\tau_s}$, $\mathcal{G}_s = \mathcal{F}_{\tau_s}$. Then (τ_s) is an $(\mathcal{F}_t)$-stopping time and $[M]_{\tau_s} = s$ for all $s \geq 0$. Moreover, B is a $(\mathcal{G}_s)$-BM with $M_t = B_{[M]_t}$.
$\Rightarrow$ every cts local MG starting from 0 is a time-change of a standard BM.

Proof: Since $[M]$ is continuous and adapted, τ_s is a stopping time for each $s \geq 0$. Since $[M]_\infty = \infty$, τ_s is a finite stopping time $\forall s \geq 0$.
Moreover, $(\mathcal{G}_s)$ is a filtration since if S, T are stopping times with $S \leq T$ then $\mathcal{F}_S \subseteq \mathcal{F}_T \Rightarrow$ hence $s \leq t \Rightarrow \mathcal{G}_s \subseteq \mathcal{G}_t$.

Step 1: B is adapted to $(\mathcal{G}_s)$
NTS: M_{τ_s} is $\mathcal{F}_{\tau_s}$-measurable $\forall s \geq 0$.
Recall from ES1, that if X is cadlag, adapted, T a stopping time then $X_T \cdot \mathbb{1}(T < \infty)$ is $\mathcal{F}_T$-meas.
Apply for $X = M$, and $T = \tau_s$ and use that $\mathbb{1}(\tau_s < \infty) = 1$.

Step 2: B is continuous.
$s \mapsto \tau_s$ is non-decreasing and cadlag, so it follows that B is cadlag $[B_s = M_{\tau_s}]$.
To prove that B is cts, NTS: $B_{s-} = B_s$ $\forall s \geq 0$.
$\iff M_{\tau_{s^-}} = M_{\tau_s}$ $\forall s \geq 0$ where
$\tau_{s^-} = \inf\{t \geq 0 : [M]_t = s\}$. If $\tau_s = \tau_{s^-}$, nothing to prove. If $\tau_s > \tau_{s^-}$, then $[M]$ is constant on $[\tau_{s^-}, \tau_s]$.
NTS: $[M]$ is constant on any interval, then M is constant as well.



for each rational $q \in \mathbb{Q}$, $S_q := \inf\{t \geq q : [M]_t > [M]_q\}$
$[q, S_q]$.

(*) let $A \in \mathcal{F}_s$, if $P(A) \neq 0$, define the prob. measure $P_A(\cdot) = P(A)^{-1}\cdot P(\cdot \cap A)$
$\Rightarrow$ (Tower) $\mathbb{E}_{P_A}[\exp(i\theta(X_t - X_s))] = \mathbb{E}_P[\exp(i\theta(X_t - X_s))]$
$\Rightarrow$ Law of $X_t - X_s$ under P_A is the same as that under P. Hence, $\forall f: \mathbb{R}^d \to \mathbb{R}$ bded measurable:
$$\mathbb{E}[\mathbb{1}_A \cdot f(X_t - X_s)] = P(A)\cdot\mathbb{E}[f(X_t - X_s)]$$
$\Rightarrow X_t - X_s \perp\!\!\!\perp \mathcal{F}_s$.

LECTURE 17

Proof: (Dubins–Schwarz)
Working on Step 2, which is the continuity of B. Need to prove that if $[M]$ is constant on a given interval, then M is constant on the same interval. By localisation, WLOG, $M \in \mathcal{M}_c^2$. Suppose that $q \in \mathbb{Q}$, $q > 0$, and let $S_q = \inf\{t \geq q : [M]_t > [M]_q\}$. Suffices to show that M is a.s. constant on each $[q, S_q]$.

We know that $M^2 - [M]$ is a UI since $M \in \mathcal{M}_c^2$. By OST, we have that:

$(\circledast)$ $\mathbb{E}[M_{S_q}^2 - [M]_{S_q} | \mathcal{F}_q] = M_q^2 - [M]_q$.

Since $M \in \mathcal{M}_c^2$, we also have that:

(MG orthog) $\mathbb{E}[(M_{S_q} - M_q)^2 | \mathcal{F}_q] = \mathbb{E}[M_{S_q}^2 - M_q^2 | \mathcal{F}_q]$

$(\circledast)$ $= \mathbb{E}[[M]_{S_q} - [M]_q | \mathcal{F}_q] = 0$ since $[M]_{S_q} = [M]_q$.

Therefore $M_{S_q} - M_q = 0$ a.s. $\Rightarrow$ M is a.s. constant on $[q, S_q]$. ($\forall t \geq q$: $M_{t \wedge S_q} = \mathbb{E}[M_{S_q} | \mathcal{F}_t]$
$= \mathbb{E}[M_q | \mathcal{F}_t] = M_q$ a.s.)

Step 3: B is a $(\mathcal{G}_s)$-BM.
Fix $s > 0$. Then we know that $[M^{\tau_s}]_\infty = [M]_{\tau_s} = s$. Therefore $M^{\tau_s} \in \mathcal{M}_c^2$. Since $\mathbb{E}[[M^{\tau_s}]_\infty] < \infty$. Therefore $(M^2 - [M])^{\tau_s}$ is a UI MG. By OST, for $0 \leq r \leq s < \infty$, we have that:

i) $\mathbb{E}[B_s | \mathcal{G}_r] = \mathbb{E}[M_\infty^{\tau_s} | \mathcal{F}_{\tau_r}] = M_{\tau_r} = B_r$.

ii) $\mathbb{E}[B_s^2 - s | \mathcal{G}_r] = \mathbb{E}[(M^2 - [M])_\infty^{\tau_s} | \mathcal{F}_{\tau_r}]$
$= M_{\tau_r}^2 - [M]_{\tau_r}$
$= B_r^2 - r$

Thus i) $\Rightarrow$ $B \in \mathcal{M}_c$
ii) $\Rightarrow$ $[B]_t = t$

$\Rightarrow$ B is a $(\mathcal{G}_s)$-BM by the Levy characterisation $\square$

Dubins–Schwarz requires $[M]_\infty = \infty$. Extension to the case that $[M]_\infty < \infty$.

Theorem: $M \in \mathcal{M}_{c,loc}$, $M_0 = 0$. Let β be a BM which is independent of M. Set:

$$B_s = \begin{cases} M_{\tau_s}, & s < [M]_\infty \\ M_\infty + \beta_{s - [M]_\infty} & s \geq [M]_\infty \end{cases}$$

Then B is a standard BM and $M_t = B_{[M]_t}$ for all $t \geq 0$.

Examples: i) B is a standard BM, h deterministic, measurable in $L^2([0,\infty))$. Let $M_t = \int_0^t h(s)\, dB_s$. Then $M_0 = 0$, $M \in \mathcal{M}_{c,loc}$

and $[M]_t = \int_0^t h^2(s)\, ds$. Moreover,
$M_\infty \overset{d}{=} B_{\int_0^\infty h^2(s) ds}$ (Dubins–Schwarz) $\sim N(0, \|h\|_{L^2}^2)$.

ii) Let $M \in \mathcal{M}_{c,loc}$. Then:
$\{[M]_\infty < \infty\} = \{\lim_{t\to\infty} M_t \text{ exists}\}$

$\{[M]_\infty = \infty\} = \{\liminf_{t\to\infty} M_t = -\infty, \limsup_{t\to\infty} M_t = \infty\}$.

(check!)

Exponential MGs: Let $M \in \mathcal{M}_{c,loc}$, $M_0 = 0$. Set $Z_t = \exp(M_t - \frac{1}{2}[M]_t)$. By Itô's formula,
$dZ_t = Z_t(dM_t - \frac{1}{2} d[M]_t) + \frac{1}{2} Z_t\, d[M]_t$
$= Z_t\, dM_t \Rightarrow Z \in \mathcal{M}_{c,loc}$, $Z_0 = 1$

Definition: (Exponential MG) In the setting above, the process $\mathcal{E}(M)_t = Z_t = \exp(M_t - [M]_t/2)$ is the *stochastic exponential* or *exponential martingale* associated with M.

Note that $\mathcal{E}(M) \in \mathcal{M}_{c,loc}$, $d\mathcal{E}(M)_t = \mathcal{E}(M)_t\, dM_t$.

Proposition: Let $M \in \mathcal{M}_{c,loc}$ $M_0 = 0$. If $[M]$ is bdd, then $\mathcal{E}(M)$ is a UI MG.

Proposition: Let $M \in \mathcal{M}_{c,loc}$, $M_0 = 0$. For all $\varepsilon, \delta > 0$, we have that $\mathbb{P}(\sup_{t\geq 0} M_t \geq \varepsilon, [M]_\infty < \delta) \leq e^{-\frac{\varepsilon^2}{2\delta}}$

Proof: Fix $\varepsilon > 0$ and let $T = \inf\{t \geq 0 : M_t \geq \varepsilon\}$. Fix $\theta > 0$ and set $Z_t = \mathcal{E}(\theta M^T)_t$
$= \exp(\theta M_t^T - \frac{1}{2}\theta^2 [M]_t^T) \in \mathcal{M}_{c,loc}$.
Note that $|Z_t| \leq e^{\theta\varepsilon}$ for all $t \geq 0$. So Z is a bdd MG $\Rightarrow$ $\mathbb{E}[Z_\infty] = Z_0 = 1$. For $\delta \geq 0$, we have that

$P[\sup_{t\geq 0} M_t \geq \varepsilon, [M]_\infty \leq \delta]$

$= \mathbb{P}[\sup_{t\geq 0} e^{\theta M_t^T} \geq e^{\theta\varepsilon}, [M]_\infty \leq \delta]$

$\leq \mathbb{P}[\sup_{t\geq 0} Z_t \geq e^{\theta\varepsilon - \theta^2\delta/2}]$.
(Doob's ineq.)
$\leq \exp[-\theta \cdot \varepsilon + \theta^2\delta/2]$.
Optimise over θ gives claimed bound. $\square$

Proof of (previous) proposition: Will show that $\mathcal{E}(M)$ is bounded by an integrable random variable. Note that:
$\sup_{t\geq 0} \mathcal{E}(M)_t \leq \exp[\sup_{t\geq 0} M_t]$ (since $[M]_t \geq 0$).

NTS: RHS is integrable. Let $C > 0$ so that $[M]_\infty \leq C$. Then: $\mathbb{P}[\sup_{t\geq 0} M_t \geq \varepsilon] = \mathbb{P}[\sup_{t\geq 0} M_t \geq \varepsilon, [M]_\infty \leq C]$

$\leq \exp[-\varepsilon^2/2C]$
$\Rightarrow \mathbb{E}[\exp[\sup_{t\geq 0} M_t]] = \int_0^\infty \mathbb{P}(\exp[\sup_{t\geq 0} M_t] \geq \lambda)\, d\lambda$.

$= \int_0^\infty \mathbb{P}[\sup_{t\geq 0} M_t \geq \log\lambda]\, d\lambda$

$\leq \underbrace{1}_{\lambda \in [0,1]} + \int_1^\infty \underbrace{\exp[-(\log\lambda)^2/2C]}_{\lambda \in [1,\infty)}\, d\lambda < \infty$

$\Rightarrow \mathcal{E}(M)$ is UI $\square$

LECTURE 18

Suppose that $\mathbb{P}, \mathbb{Q}$ are probability measures on $(\Omega, \mathcal{F})$. Say that $\mathbb{Q}$ is <u>absolutely continuous</u> wrt $\mathbb{P}$ denoted by $\mathbb{Q} \ll \mathbb{P}$ if for any $A \in \mathcal{F}$ with $\mathbb{P}[A] = 0 \Rightarrow \mathbb{Q}[A] = 0$.

Radon-Nikodym $\Longrightarrow$ $\mathbb{Q} \ll \mathbb{P} \Rightarrow \exists$ a random variable $Z \geq 0$ such that $\mathbb{Q}[A] = \mathbb{E}_{\mathbb{P}}[Z \cdot 1_A]$ for all $A \in \mathcal{F}$. Z is called the Radon-Nikodym derivative of $\mathbb{Q}$ wrt $\mathbb{P}$ and is denoted by $Z = d\mathbb{Q}/d\mathbb{P}$.

<u>Example</u>: Suppose that $X \sim N(0,1)$, $\mu \in \mathbb{R}$. Let $Z = \exp[\mu X - \mu^2/2]$. Then $A \mapsto \mathbb{E}[1_A Z]$ defines a probability measure $\mathbb{Q}$ and under $\mathbb{Q}$, $X \sim N(\mu, 1)$.

The Girsanov Theorem generalises this idea to the setting of semiMGs, except instead of changing the mean we will change the semiMG decomposition.

<u>Theorem</u>: (Girsanov) $M \in \mathcal{M}_{c,loc}$, $M_0 = 0$, and assume that $Z = \mathcal{E}(M)$ is UI. Then we can construct a new probability measure $\tilde{\mathbb{P}} \ll \mathbb{P}$ on $(\Omega, \mathcal{F})$ by setting $\tilde{\mathbb{P}}[A] = \mathbb{E}[Z_\infty \cdot \mathbb{1}(A)]$ $\forall A \in \mathcal{F}$. If $X \in \mathcal{M}_{c,loc}(\mathbb{P})$, then $X - [X, M] \in \mathcal{M}_{c,loc}(\tilde{\mathbb{P}})$.

"Change of measure induces a change of drift".

<u>Proof</u>: Since Z is UI, we know that Z_∞ exists and $Z_\infty \geq 0$ with $\mathbb{E}[Z_\infty] = 1 \Longrightarrow \tilde{\mathbb{P}}$ defines a probability measure with $\tilde{\mathbb{P}} \ll \mathbb{P}$. Suppose that $X \in \mathcal{M}_{c,loc}(\mathbb{P})$ and set $T_n := \inf\{t \geq 0 : |X_t - [X, M]_t| \geq n\}$. Since $X - [X, M]$ is continuous (starts from zero), we have that: $\mathbb{P}[T_n \uparrow \infty] = 1 \Rightarrow \tilde{\mathbb{P}}[T_n \uparrow \infty] = 1$ (since $\tilde{\mathbb{P}} \ll \mathbb{P}$). To prove that $Y = X - [X, M] \in \mathcal{M}_{c,loc}(\tilde{\mathbb{P}})$, it suffices to show that $Y^{T_n} = X^{T_n} - [X^{T_n}, M] \in \mathcal{M}_c(\tilde{\mathbb{P}})$ $\forall n$. In what follows, write <u>X, Y in place of X^{T_n}, Y^{T_n}</u>.

$$
\begin{aligned}
d(Z_t Y_t) &\overset{\text{IBP}}{=} Y_t dZ_t + Z_t dY_t + dY_t dZ_t \quad \text{(IBP)}. \\
&= Y_t dZ_t + Z_t dX_t - Z_t dX_t dM_t + dX_t dZ_t \\
&= Y_t dZ_t + Z_t dX_t \quad (dZ_t = Z_t dM_t).
\end{aligned}
$$

$\Longrightarrow ZY \in \mathcal{M}_{c,loc}(\mathbb{P})$.

Moreover, $\{Z_T : T \leq t$ is a stopping time$\}$ is UI for each $t > 0$ (Ex. Sheet 1). Since Y is bounded, we also have that:

$\{Z_T Y_T : T \leq t$ is a stopping time$\}$ is also UI $\Longrightarrow ZY \in \mathcal{M}_c(\mathbb{P})$. For $s < t$, we have that $\tilde{\mathbb{E}}[Y_t - Y_s | \mathcal{F}_s] = 1/Z_s \cdot \mathbb{E}[Z_\infty Y_t - Z_\infty Y_s | \mathcal{F}_s]$

$$
\text{(Tower property)} = \frac{1}{Z_s} \mathbb{E}[Z_t Y_t - Z_s Y_s | \mathcal{F}_s] = 0
$$

since $ZY \in \mathcal{M}_c(\mathbb{P}) \Longrightarrow Y \in \mathcal{M}_c(\tilde{\mathbb{P}})$. $\square$

<u>Remark</u>: The quadratic variation does not change when performing a change of measures (ES4).

<u>Corollary</u>: Let B be a standard Brownian motion under $\mathbb{P}$, $M \in \mathcal{M}_{c,loc}$, $M_0 = 0$. Suppose that $Z = \mathcal{E}(M)$ is UI and $\tilde{\mathbb{P}}(A) = \mathbb{E}[1_A Z_\infty]$ for all $A \in \mathcal{F}$. Then $\tilde{B} = B - [B, M]$ is a $\tilde{\mathbb{P}}$ Brownian motion.

<u>Proof</u>: Since $\tilde{B} \in \mathcal{M}_{c,loc}(\tilde{\mathbb{P}})$ by the Girsanov theorem, and $[\tilde{B}]_t = [B]_t = t$, it follows from the Lévy characterisation that $\tilde{B}$ is a $\tilde{\mathbb{P}}$ - Brownian motion. $\square$

<u>Example</u>: Suppose that B is a $\mathbb{P}$-Brownian motion, $\mu \in \mathbb{R}$, $T > 0$, and let $M_t = \mu B_t^T$ so that $Z_t = \mathcal{E}(M)_t = \exp[\mu B_t^T - \mu^2 t \wedge T/2]$.
$\tilde{\mathbb{P}}(A) = \mathbb{E}[Z_\infty \cdot 1_A] = \mathbb{E}[\exp[\mu B_T - \mu^2 T/2] \cdot 1_A]$ $\forall A \in \mathcal{F}$. Then under $\tilde{\mathbb{P}}$, $B_t = \tilde{B}_t + \mu t$ for $t \in [0, T]$ and $\tilde{B}$ is a $\tilde{\mathbb{P}}$ - Brownian motion.

<u>Stochastic Differential Equations:</u>

Let $M^{d \times m}(\mathbb{R})$ denote the space of $d \times m$ matrices with real entries. Suppose that $\sigma : \mathbb{R}^d \to M^{d \times m}(\mathbb{R})$ and $b : \mathbb{R}^d \to \mathbb{R}^d$ are measurable functions which are bounded on compact sets. Write $\sigma(x) = (\sigma_{ij}(x))$, $b(x) = (b_i(x))$. Consider: $dX_t = \sigma(X_t) dB_t + b(X_t) dt$. ($\ast$)
Equivalently, $dX_t^i = \sum_{j=1}^m \sigma_{ij}(X_t) dB_t^j + b_i(X_t) dt$

A solution to ($\ast$) consists of:

- A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ where $(\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions.
- An $(\mathcal{F}_t)_{t \geq 0}$ - Brownian motion $B = (B^1, \dots, B^m) \in \mathbb{R}^m$.
- An $(\mathcal{F}_t)_{t \geq 0}$ - adapted, continuous process $X = (X^1, \dots, X^d)$ in $\mathbb{R}^d$ such that

$$
X_t = X_0 + \int_0^t \sigma(X_s) dB_s + \int_0^t b(X_s) ds
$$

When in addition, $X_0 = x \in \mathbb{R}^d$, we say that X is started from x.

- We say that an SDE has a <u>weak solution</u> if for all $x \in \mathbb{R}^d$, there is a solution starting from x.
- There is <u>uniqueness in law</u> if all solutions starting from each x have the same distribution.
- There is <u>pathwise uniqueness</u> if when we fix $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and B then any two solutions X, X' with $X_0 = X_0'$ are indistinguishable, $\mathbb{P}(X_t = X_t' \ \forall t) = 1$.
- We say that a solution started from x is a <u>strong solution</u> if X is adapted to the filtration generated by B.

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Example: It is possible to have the existence of a weak solution and uniqueness in law without having pathwise uniqueness.

Suppose that β is a standard Brownian motion in $\mathbb{R}$ with $\beta_0 = X$. Set $B_t = \int_0^t \operatorname{sgn}(\beta_s)\, d\beta_s$, $\operatorname{sgn}(x) = 1_{(0,\infty)}(x) - 1_{(-\infty,0]}(x)$.

Note that $\operatorname{sgn}(\beta_s)$ is previsible and bounded $\Rightarrow$ integral is well-defined.

$$x + \int_0^t \operatorname{sign}(\beta_s)\, dB_s = x + \int_0^t (\operatorname{sgn}(\beta_s))^2\, d\beta_s$$

$$= x + \int_0^t d\beta_s = \beta_t$$

$\Rightarrow \beta$ solves the SDE $\begin{cases} dX_t = \operatorname{sgn}(X_t)\, dB_t \\ X_0 = x \end{cases}$

This SDE has a weak solution. By the Lévy characterisation, any solution to this SDE is a Brownian motion. [It is an $\mathcal{M}_{c,loc}$ with quadratic variation $= t$.] $\Rightarrow$ we have uniqueness in law.

We do not have pathwise uniqueness.

To see this, take $X_0 = x = 0$.

Claim: $\beta, -\beta$ are solutions. *Know:* β is a solution.

$$-\beta_t = -\int_0^t \operatorname{sign}(\beta_s)\, d\beta_s = \int_0^t \operatorname{sign}(-\beta_s)\, dB_s + 2\int_0^t 1\{\beta_s = 0\}\, dB_s.$$

The last term on RHS is in $\mathcal{M}_{c,loc}$, starts from 0, and has quadratic variation:

$4\int_0^t 1\{\beta_s = 0\}\, ds = 0$ a.s.

[its expectation is $= 0$ since $\mathbb{P}[\beta_s = 0] = 0$ $\forall s > 0$ and apply Fubini's theorem].

Therefore $\beta, -\beta$ are both solutions on the same probability space with the same Brownian motion. So we cannot have pathwise uniqueness.

Lipschitz coefficients:

Recall that for $U \subseteq \mathbb{R}^d$ open, $f: U \to \mathbb{R}^d$ we say that f is Lipschitz if there exists $K < \infty$ so that $|f(x) - f(y)| \le K \cdot |x-y|$ for all $x, y \in U$.

For $\dim \ge 1$ we equip $M^{d\times m}(\mathbb{R})$ with the Frobenius norm. $A \in M^{d\times m}(\mathbb{R})$, $A = (a_{ij})$, $|A| = \left(\sum_{i=1}^d \sum_{j=1}^m a_{ij}^2\right)^{1/2}$. If $f: U \to M^{d\times m}(\mathbb{R})$, say that f is Lipschitz if there exists $K < \infty$ so that $|f(x) - f(y)| \le K \cdot |x-y|$ $\forall x, y \in U$.

__Theorem__ (Existence and uniqueness) Suppose that $\sigma: \mathbb{R}^d \to M^{d\times m}(\mathbb{R})$, $b: \mathbb{R}^d \to \mathbb{R}^d$ are Lipschitz. There is pathwise uniqueness for the SDE $dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt$. Moreover, for each filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ satisfying the usual conditions and each $(\mathcal{F}_t)$-Brownian motion B, $x \in \mathbb{R}^d$ there is a strong solution starting from x.

__Proof__ is analogous to the existence/uniqueness for ODEs.

__Theorem:__ Let (X, d) be a complete metric space.

a) Suppose that $F: X \to X$ is a contraction, i.e., $\exists\, r \in (0,1)$ s.t. $d(F(x), F(y)) \le r \cdot d(x,y)$ $\forall x, y \in X$. Then F has a unique fixed point.

b) Suppose that $F: X \to X$ and there exists $n \in \mathbb{N}$ so that $F^{(n)}$ is a contraction. Then F has a unique fixed point.

__Lemma:__ (Gronwall) Let $T > 0$ and $f: [0,T] \to [0,\infty)$ be a bounded and measurable function. If there exist $a, b \ge 0$ s.t. $f(t) \le a + b\int_0^t f(s)\, ds$ $\forall t \in [0,T]$, then $f(t) \le a \cdot e^{bt}$ $\forall t \in [0,T]$.

__Proof:__ ES3.

__Proof of existence and uniqueness:__ Will assume that $\dim = 1$. Let K be such that $|\sigma(x) - \sigma(y)| \le K \cdot |x-y|$, $|b(x) - b(y)| \le K \cdot |x-y|$ $\forall x, y \in \mathbb{R}$.

__Proof of uniqueness:__ Suppose that X, X' are two solutions on the same probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ and Brownian motion B.

__WTS:__ $\mathbb{P}(X_t = X_t'\ \forall t \ge 0) = 1$. Fix $M > 0$ and let $\tau = \inf\{t \ge 0 : |X_t| \vee |X_t'| \ge M\}$.

Then: $\begin{cases} X_{t\wedge\tau} = X_0 + \int_0^{t\wedge\tau} \sigma(X_s)\, dB_s + \int_0^{t\wedge\tau} b(X_s)\, ds \\ X'_{t\wedge\tau} = X_0 + \int_0^{t\wedge\tau} \sigma(X_s')\, dB_s + \int_0^{t\wedge\tau} b(X_s')\, ds \end{cases}$

Fix $T > 0$. If $t \in [0,T]$ we have that

$$\mathbb{E}[(X_{t\wedge\tau} - X'_{t\wedge\tau})^2] \le 2\cdot \mathbb{E}\left[\left(\int_0^{t\wedge\tau} (\sigma(X_s) - \sigma(X_s'))\, dB_s\right)^2\right] + 2\cdot \mathbb{E}\left[\left(\int_0^{t\wedge\tau} (b(X_s) - b(X_s'))\, ds\right)^2\right]$$

$\le 2\cdot \mathbb{E}\left[\int_0^{t\wedge\tau} (\sigma(X_s) - \sigma(X_s'))^2\, ds\right]$ (Itô isometry)

$+ 2\cdot T \cdot \mathbb{E}\left[\int_0^{t\wedge\tau} (b(X_s) - b(X_s'))^2\, ds\right]$ (Cauchy-Schwarz)

$\le 2\cdot K^2 \cdot (1+T) \cdot \mathbb{E}\left[\int_0^{t\wedge\tau} (X_s - X_s')^2\, ds\right]$

$\le 2\cdot K^2 \cdot (1+T) \cdot \int_0^t \mathbb{E}[(X_{s\wedge\tau} - X'_{s\wedge\tau})^2]\, ds$

Let $f(t) = \mathbb{E}[(X_{t\wedge\tau} - X'_{t\wedge\tau})^2]$. Then:

$0 \le f \le 4\cdot M^2$ and $f(t) \le 2\cdot K^2 (1+T) \int_0^t f(s)\, ds$ $\forall t \in [0,T]$. Gronwall $\Rightarrow f(t) = 0$ $\forall t \in [0,T]$.

$\Rightarrow \mathbb{P}[X_{t\wedge\tau} = X'_{t\wedge\tau}\ \forall t \in [0,T]] = 1$

M, T arbitrary $\Rightarrow \mathbb{P}[X_t = X_t'\ \forall t \ge 0] = 1$.

Pathwise uniqueness.

LECTURE 20

Proof: (Existence and uniqueness of solutions)

Existence: Suppose that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a filtered probability space, B is an $(\mathcal{F}_t)$-Brownian motion, and $(\mathcal{F}_t^B)_{t\geq 0}$ the filtration generated by B (so that $\mathcal{F}_t^B \subseteq \mathcal{F}_t$).

Will use the contraction mapping theorem. Need to specify 1) the space, 2) the map.

For each $T>0$, let $\mathcal{C}_T = \{$continuous, adapted processes $X : [0,T] \to \mathbb{R}$, $|||X|||_T = \| \sup_{0\leq t\leq T} X_t \|_{L^2} < \infty\}$

Proved before that $\mathcal{C}_T$ is complete. Fix $x \in \mathbb{R}$, using that σ, b are Lipschitz, we have that:

(1) $|\sigma(y)| = |\sigma(y) - \sigma(0) + \sigma(0)| \leq |\sigma(y) - \sigma(0)| + |\sigma(0)|$
$\leq |\sigma(0)| + K\cdot|y|$ and

(2) $|b(y)| \leq |b(0)| + K\cdot|y|$ for all $y \in \mathbb{R}$. Fix $T>0$ and $X \in \mathcal{C}_T$. Let $M_t = \int_0^t \sigma(X_s)\,dB_s$ for $0 \leq t \leq T$.

Then $[M]_T = \int_0^T \sigma^2(X_s)\,ds$. Thus

(1) $\Rightarrow \mathbb{E}[[M]_T] \leq 2\cdot T\cdot(|\sigma(0)|^2 + K^2\cdot|||X|||_T^2) < \infty$

$\Rightarrow M^T \in \mathcal{M}_c^2$, so Doob's inequality

$\Rightarrow \mathbb{E}[\sup_{0\leq t\leq T} (\int_0^t \sigma(X_s)\,dB_s)^2]$

$$\leq 8\cdot T\cdot(|\sigma(0)|^2 + K^2\cdot|||X|||_T^2).$$

By (2), $\mathbb{E}[\sup_{0\leq t\leq T} |\int_0^t b(X_s)\,ds|^2]$

$$\leq T\cdot\mathbb{E}[\int_0^T b^2(X_s)\,ds] \quad \text{(Cauchy-Schwarz)}$$
$$\leq 2T^2(|b(0)|^2 + K^2\cdot|||X|||_T^2) < \infty$$

The map F on $\mathcal{C}_T$ defined by $X \mapsto F(X)$,

$$F(X)_t = x + \int_0^t \sigma(X_s)\,dB_s + \int_0^t b(X_s)\,ds$$

takes values in $\mathcal{C}_T$.

Suppose that $X, Y \in \mathcal{C}_T$. For $0 \leq t \leq T$, similar arguments

$$|||F(X) - F(Y)|||_t^2 \leq \underbrace{4K^2T\cdot(4+T)}_{C_T} \int_0^t |||X-Y|||_s^2\,ds$$
$$= C_T \int_0^t |||X-Y|||_s^2\,ds$$

Iterate n times:

$$|||F^{(n)}(X) - F^{(n)}(Y)|||_T^2 \leq C_T^n \int_0^T \int_0^{t_1} \cdots \int_0^{t_{n-1}} |||X-Y|||_{t_n}^2\,dt_n \cdots dt_1$$

$$\leq \frac{C_T^n T^n}{n!} |||X-Y|||_T \quad (3)$$

Take n to be sufficiently large so that $C_T^n T^n / n! < 1$. Contraction mapping theorem $\Rightarrow$ there exist a unique fixed point $X^{(T)} \in \mathcal{C}_T$ of f.

Pathwise uniqueness $\Rightarrow X_t^{(T)} = X_t^{(T')}$ $\forall t \leq T \wedge T'$ a.s.
Define X by setting $X_t = X_t^{(N)}$ where $t \leq N$, $N \in \mathbb{N}$. $\Rightarrow X$ is the pathwise unique solution to the SDE starting from x.

NTS: X is a strong solution, i.e. X is adapted to $(\mathcal{F}_t^B)$. Will prove that for each fixed T, $X^{(T)}$ is the limit of $(\mathcal{F}_t^B)$-processes. Define (Y^n) in $\mathcal{C}_T$ by setting $Y^0 = x$ and $Y^n = F(Y^{n-1})$ for each $n \geq 1$. Then Y^n is adapted to $(\mathcal{F}_t^B)$ for each n. As $F^{(n)}(X) = X$. For all $n \geq 1$ we have from (3) that:

$$|||X - Y^n|||_T^2 = |||F^{(n)}(X) - F^{(n)}(x)|||_T^2$$
$$\leq \frac{C_T^n T^n}{n!} \cdot |||X - x|||_T^2 \to 0 \text{ as } n \to \infty.$$

Thus $Y^n \to X$ in $\mathcal{C}_T$ as $n \to \infty$.
$\Rightarrow \exists$ subsequence (Y^{n_k}) so that $Y^{n_k} \to X$ uniformly in $[0,T]$ a.s. Therefore (X_t^T) is the a.s. limit of $(\mathcal{F}_t^B)$-adapted processes and so is $(\mathcal{F}_t^B)$-adapted. Since $T>0$ was arbitrary, we have that X is $(\mathcal{F}_t^B)$-adapted.

$\square$

Remark: Proof $\Rightarrow$ pathwise unique strong solution lies in $\mathcal{C}_T$ for all $T>0$.

Proposition: Under the hypotheses of the theorem, there is uniqueness in law for the SDE

$$dX_t = \sigma(X_t)\,dB_t + b(X_t)\,dt.$$

Proof: Ex. Sheet 3.

Example: (Ornstein-Uhlenbeck process)
Fix $\lambda \in \mathbb{R}$ and consider the SDE

$$dV_t = dB_t - \lambda V_t\,dt, \quad V_0 = v_0$$
$$dX_t = V_t\,dt$$

For $\lambda>0$, models the movement of a grain of pollen in liquid; X = position of the grain, V = velocity. This term $-\lambda V$ dampens the system due to viscosity. $|V|$ large, the system moves to reduce $|V|$.

Theorem $\Rightarrow \exists!$ strong solution. Can explicitly solve

$$d(e^{\lambda t}V_t) = e^{\lambda t}dV_t + \lambda e^{\lambda t}V_t\,dt$$
$$= e^{\lambda t}\,dB_t$$
$$\Rightarrow e^{\lambda t}V_t = v_0 + \int_0^t e^{\lambda s}\,dB_s$$
$$\Rightarrow V_t = e^{-\lambda t}v_0 + \int_0^t e^{\lambda(s-t)}\,dB_s.$$
$$V_t \sim N\left(e^{-\lambda t}v_0, \frac{1-e^{-2\lambda t}}{2\lambda}\right).$$

If $\lambda>0$, $V_t \xrightarrow{d} N(0, (2\lambda)^{-1})$ as $t \to \infty$.
$\Rightarrow N(0, (2\lambda)^{-1})$ is the stationary distribution of V, i.e. if $V_0 \sim N(0, (2\lambda)^{-1})$ then
$V_t \sim N(0, (2\lambda)^{-1})$ $\forall t \geq 0$.

LECTURE 21

Local Solutions: $\quad dX_t = \sigma(X_t)\,dB_t + b(X_t)\,dt.$

A *locally defined process* is a pair (X, τ) consisting of a stopping time τ together with a map $X : \{(\omega, t) \in \Omega \times [0,\infty) : t < \tau(\omega)\} \to \mathbb{R}$. It is *cadlag* if the map $t \mapsto X_t(\omega)$ from $[0, \tau(\omega)) \to \mathbb{R}$ is cadlag for all $\omega \in \Omega$. Let $\Omega_t = \{\omega \in \Omega : t < \tau(\omega)\}$. Then (X, τ) is *adapted* if $X_t : \Omega_t \longrightarrow \mathbb{R}$ is $\mathcal{F}_t$-measurable.
We say that (X, τ) is a *locally defined MG* if there exist stopping times $T_n \uparrow \tau$ so that X^{T_n} is a MG for all n. We say that (H, η) is a *locally defined locally bounded previsible process* if there exist stopping times $S_n \uparrow \eta$ such that $H \cdot 1\!\!1_{(0,S_n]}$ is bounded and previsible $\forall n \in \mathbb{N}$.
We define:

$(H \cdot X, \tau \wedge \eta)$ by setting

$$(H \cdot X)_t^{T_n \wedge S_n} = (H 1\!\!1_{(0, S_n \wedge T_n]} \cdot X^{S_n \wedge T_n})_t$$

for each n.

Proposition: (Local Itô's formula) Let $X^1, \dots, X^d$ be cont. semiMGs, let $U \subseteq \mathbb{R}^d$ be open, and let $f : U \longrightarrow \mathbb{R}$ be C^2. Let $X = (X^1, \dots, X^d)$ and set $\tau = \inf\{t \geq 0 : X_t \notin U\}$. Then for all $t < \tau$ we have that

$$f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(X_s)\,dX_s^i + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\partial^2 f(X_s)}{\partial x_i \partial x_j}\,d[X^i, X^j]_s$$

Proof: Apply Itô's formula to X^{T_n} where
$T_n = \inf\{t \geq 0 : \text{dist}(X_t, U^c) \leq 1/n\}$
and note that $T_n \uparrow \tau$ as $n \longrightarrow \infty$. $\square$

Example: let $X = B$ where B is a standard Brownian motion with $X_0 = B_0 = 1$, $U = (0, \infty)$, $f(x) = \sqrt{x}$. Then

$$\sqrt{B_t} = 1 + \frac{1}{2}\int_0^t B_s^{-1/2}\,dB_s - \frac{1}{8}\int_0^t B_s^{-3/2}\,ds \text{ for}$$

all $t < \tau = \inf\{t \geq 0 : B_t = 0\}$.

Let $U \subseteq \mathbb{R}^d$ be open, $\sigma : U \to M^{d\times m}(\mathbb{R})$, $b : U \to \mathbb{R}^d$ be measurable functions which are bounded on compact subset of U.

A *local solution* to the SDE

$$dX_t = \sigma(X_t)\,dB_t + b(X_t)\,dt$$

consists of:

- A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions.
- An $(\mathcal{F}_t)$ - Brownian motion B in $\mathbb{R}^m$.
- A continuous $(\mathcal{F}_t)$-adapted locally defined process (X, τ)

with $X \in \mathbb{R}^d$ such that:

$$X_t = X_0 + \int_0^t \sigma(X_s)\,dB_s + \int_0^t b(X_s)\,ds$$

for all $t < \tau$.

We say that (X, τ) is a *maximal local solution* if for any other local solution (Y, η) on the same space such that $X_t = Y_t$ for all $t \leq \tau \wedge \eta$ we have that $\eta \leq \tau$.

Locally Lipschitz coefficients. Suppose that $U \subseteq \mathbb{R}^d$ is open. Then a function $f : U \to \mathbb{R}^d$ is *locally Lipschitz* if for each $C \subseteq U$ compact we have that $f|_C$ is Lipschitz.

Theorem: Suppose $U \subseteq \mathbb{R}^d$ is open and $\sigma : U \to M^{d\times m}(\mathbb{R})$, $b : U \to \mathbb{R}^d$ are locally Lipschitz. Then for all $x \in U$, the PDE $dX_t = \sigma(X_t)\,dB_t + b(X_t)\,dt$ has a pathwise unique maximal local solution (X, τ) starting from x. Moreover, for all compact sets $C \subseteq U$, on the event that $\{\tau < \infty\}$ we have that $\sup\{t < \tau : X_t \in C\} < \tau$.

Lemma: Let $U \subseteq \mathbb{R}^d$ be open, $C \subseteq U$ be compact. Then: i) There exists a C^∞ function $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that $\varphi|_C \equiv 1$ and $\varphi|_{U^c} \equiv 0$.
ii) Given a locally Lipschitz function $f : U \to \mathbb{R}$, then there exists a globally Lipschitz function $g : \mathbb{R}^d \to \mathbb{R}$ such that $f|_C = g|_C$.

Proof: i) Exercise ii) φ as in i) and set $g = f \cdot \varphi$. $\square$

Proof: (theorem)
Assume that $d = m = 1$. Fix $C \subseteq U$ be compact. By the lemma, we can find Lipschitz functions $\tilde\sigma, \tilde b$ on $\mathbb{R}$ such that $\tilde\sigma|_C = \sigma|_C$, $\tilde b|_C = b|_C$.
There exists a pathwise unique strong solution $\tilde X$ to:

$$\begin{cases} d\tilde X_t = \tilde\sigma(\tilde X_t)\,dB_t + \tilde b(\tilde X_t)\,dt \\ \tilde X_0 = x \end{cases}$$

Let $T = \inf\{t \geq 0 : \tilde X_t \notin C\}$ and let $X = \tilde X|_{[0,T)}$.
Then (X, T) is a local solution in C. (check)

If $T < \infty$, then $X_{T-} = \lim_{s \uparrow T} X_s$ exists and is in $U \setminus C^0$. Suppose that $(X, T), (Y, S)$ are both local solutions in C. Let:

$$f(t) = \mathbb{E}\Big[\sup_{0 \leq s \leq S \wedge T \wedge t} |X_s - Y_s|^2\Big]$$

As b, σ are Lipschitz on C, we can use Gronwall's lemma as before to see that $f \equiv 0$
$\Longrightarrow X_t = Y_t \quad \forall t \leq S \wedge T$ a.s.

Let (C_n) be a sequence of compact sets in U with $C_n \subseteq C_{n+1}$ for all n, $U = \bigcup_n C_n$. Let (X^n, T_n) be the local solution constructed above $C = C_n$. If $T_n < \infty$, $X^n_{T_n-} \in U \setminus C_n^0$. Since the are non-decreasing, we have $T_n \uparrow \tau$, $\tau = \sup_n T_n$. (*)
Define the local solution by setting $X_t = X_t^n$ for all $t < T_n$. Consistent by the above.
NTS: (X, τ) is maximal.

(*) On $\underbrace{\inf\{t \geq 0 : X_t^{n+1} \notin C_n^0\}}_{\tilde T_n} \wedge T_n := S_n$

$X^{n+1}_{t \wedge S_n} = X^n_{t \wedge S_n}$ a.s. $\forall t \leq S_n$
(Gronwall-type argument). Suppose for a contradiction that $\tilde T_n < T_n$. Then the above
$\Longrightarrow X_t^{n+1} = X_t^n$ a.s. $\forall t \leq \tilde T_n$, giving
$U \setminus C_n^0 \ni X^{n+1}_{\tilde T_n -} = X^n_{\tilde T_n -} \in C_n^0$ ↯. Hence
$T_n \leq \tilde T_n \leq T_{n+1} \Rightarrow (T_n)$ is increasing.

LECTURE 22

Proof: left to show (1) maximality, (2) $\sup\{t<\tau : X_t \in C\} < \tau$ on $\{\tau<\infty\}$.

Suppose that (Y,η) is another solution on the same probability space. For each n, set $S_n = \inf\{t \le \eta : Y_t \notin C_n\} \wedge n$. By the uniqueness of the solution in each C_n, we have that $X_t = Y_t$ $\forall t \le S_n \wedge T_n$
$\Rightarrow S_n \le T_n$. (*) As $n \to \infty$, $S_n \uparrow \eta$, $T_n \uparrow \tau$
$\Rightarrow \eta \le \tau \Rightarrow X_t = Y_t$ $\forall t \le \eta$.
Therefore (X,τ) is maximal.

Suppose that C_1, C_2 are compact sets in U with $C_1 \subseteq C_2^{\circ} \subseteq C_2 \subseteq U$. Let $\varphi : U \to \mathbb{R}$ be a C^∞ function with $\varphi|_{C_1} \equiv 1$, $\varphi|_{(C_2^\circ)^c} \equiv 0$.

Let
$$R_0 = \inf\{t \ge 0 : X_t \notin C_2\}$$
$$S_n = \inf\{t \ge R_{n-1} : X_t \in C_1\} \wedge \tau$$
$$R_n = \inf\{t \ge S_n : X_t \notin C_2\} \wedge \tau.$$

Let N be the # of crossings that X makes from C_2 to C_1. On $\{\tau \le t, N \ge n\}$, we have that:
$$\sum_{k=1}^n \big(\underbrace{\varphi(X_{R_k})}_{=0} - \underbrace{\varphi(X_{S_k})}_{=1}\big) = -n$$
$$= \int_0^t \sum_{k=1}^n \mathbb{1}_{(S_k,R_k]}(s)\cdot\left(\varphi'(X_s)\,dX_s + \tfrac{1}{2}\varphi''(X_s)\,d[X]_s\right)$$
$$= \int_0^t H_s^n\,dB_s + K_s^n\,ds =: Z_t^n$$
where H^n, K^n are previsible and bounded uniformly in n. Then:
$$n^2\,\mathbb{1}\{\tau \le t, N \ge n\} \le (Z_t^n)^2$$
$$\Rightarrow \mathbb{P}[\tau \le t, N \ge n] \le \frac{1}{n^2}\mathbb{E}[(Z_t^n)^2]$$
Since H^n, K^n are uniformly bounded and Z_t^n is defined by integrating H^n, K^n over a time-interval which does not depend on n. We have that $\mathbb{E}[(Z_t^n)^2] \le C$ where C does not depend on n.
$$\Longrightarrow \mathbb{P}[\tau \le t, N \ge n] \le C/n^2$$
$$N\to\infty \Longrightarrow \mathbb{P}[\tau \le t, N = \infty] = 0.$$
$$t \uparrow \infty \Rightarrow \mathbb{P}[\tau < \infty, N = \infty] = 0$$
$\Rightarrow$ # crossings that X makes from C_2 to C_1 is $<\infty$ on $\tau<\infty$ a.s.

Since each crossing that X makes from C_1 to C_2 a.s. takes a positive amount of time, by continuity.
$\Longrightarrow \sup\{t<\tau : X_t \in C_1\} < \tau$ on $\{\tau<\infty\}$.
$(C_1 \subset\subset U)$.

Example: (Bessel processes) Fix $\nu \in \mathbb{R}$ and consider the SDE in $U=(0,\infty)$ given by:
$$dX_t = dB_t + \frac{\nu-1}{2X_t}\,dt, \quad X_0 = x_0 \in U.$$
Thm $\Rightarrow \exists!$ maximal local solution (X,τ) in U and $\tau = \inf\{t \ge 0 : X_t = 0\}$.
(X,τ) is a <u>Bessel process of dimension ν</u>.

Suppose that $\nu \in \mathbb{N}$, β is a Brownian motion in $\mathbb{R}^\nu$ with $|\beta_0| = x_0 > 0$. Set $Y_t = |\beta_t|$ and $\eta = \inf\{t \ge 0 : B_t = 0\}$.
By the local Itô formula, we have that
$$dY_t = \frac{(\beta_t, d\beta_t)}{|\beta_t|} \leftarrow \text{inner product.} + \frac{\nu-1}{2|\beta_t|}\,dt, \quad t<\eta.$$
where $(\cdot,\cdot)$ = Euclidean inner product.
Then the process $W_t = \int_0^t \frac{(\beta_s, d\beta_s)}{|\beta_s|}$ is in $\mathcal{M}_{c,loc}$.
Moreover,
$$d[W_t] = \frac{1}{|\beta_t|^2}\sum_{i,j=1}^\nu \beta_t^i \beta_t^j\, d[\beta^i,\beta^j]_t = dt$$
Lévy characterisation $\Longrightarrow W$ is a standard BM.
$$dY_t = dW_t + \frac{\nu-1}{2Y_t}\,dt, \quad t<\eta.$$
A Bessel process of dimension ν describes the time evolution of the norm of a ν-dim. Brownian motion up to when it first hits 0.

Diffusion processes: Suppose that $a: \mathbb{R}^d \to M^{d\times d}(\mathbb{R})$, $b: \mathbb{R}^d \to \mathbb{R}^d$ are bounded, measurable, a is symmetric (i.e. $a(x)$ is symmetric for each x).
For $f \in C_b^2(\mathbb{R}^d)$ [C^2 with bdd derivatives], set
$$Lf(X) = \frac{1}{2}\sum_{i,j} a_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^d b_i(x)\frac{\partial f}{\partial x_i}(x).$$

Let X be continuous, adapted process in $\mathbb{R}^d$. Say that X is an L-diffusion if for all $f \in C_b^2(\mathbb{R}^d)$ we have that:
$$M_t^f := f(X_t) - f(X_0) - \int_0^t Lf(X_s)\,ds \text{ is a MG.}$$
(coefficient a is called the <u>diffusion</u>, and b is the <u>drift</u>).

Example: σ, b constant and $a = \sigma\sigma^T$. B standard BM on $\mathbb{R}^d$. Then
$$X_t = \sigma B_t + bt \text{ is an } (a,b)\text{-diffusion.}$$
$\sigma = I$, $b = 0$, $X_t = B_t$ is an L-diffusion where $L = \frac{1}{2}\Delta$.

Proposition: Suppose that X solves
$$dX_t = \sigma(X_t)\,dB_t + b(X_t)\,dt$$
Let $f \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ [bounded derivatives, C^1 in the first variable, C^2 in the second variable].
Then, $M_t^f = f(t,X_t) - f(0,X_0) - \int_0^t \left(\frac{\partial}{\partial s} + L\right) f(s,X_s)\,ds$ is in $\mathcal{M}_{c,loc}$, $a = \sigma\sigma^T$ and L is as above.
If a, b are bounded, then X is an L-diffusion.

Proof: ES3.

(*) Suppose for a contradiction $T_n < S_n \Rightarrow C_n \ni Y_{T_n} = X_{T_n} \in U \setminus C_n^{\circ}$ ↯

LECTURE 23

Question: Which a can be written as $\sigma\sigma^T$ for such σ? (See proposition from last time).

Suppose that a, b are Lipschitz, bounded and there exists $\varepsilon > 0$ so that:

$$(a(x)\xi, \xi) \geq \varepsilon |\xi|^2 \quad \forall x, \xi \in \mathbb{R}^d.$$

Then a is <u>uniformly positive definite</u> (UPD).

Then there exists $\sigma : \mathbb{R}^d \to M^{d\times d}(\mathbb{R})$ with $\sigma\sigma^T = a$.

For $d = 1$, take $\sigma = \sqrt{a}$.

For $d \geq 2$, we can write $a(x) = U(x)\lambda(x)U^T(x)$ where $\lambda(x)$ is the diagonal matrix of evals and $U(x)$ the orthogonal matrix whose columns are eigenvectors of $a(x)$.

Take: $\sigma(x) = U(x)\sqrt{\lambda(x)}\,U^T(x)$.

That σ is Lipschitz follows from the differentiability of the square root map on the set of UPD matrices.

For such σ, b, the SDE

$$dX_t = \sigma(X_t)\,dB_t + b(X_t)\,dt$$

has a unique strong solution which is an (a, b)-diffusion.

Proposition: Let X be an L-diffusion and T a finite stopping time. Set $\tilde{X}_t = X_{T+t}$ and $\tilde{\mathcal{F}}_t = \mathcal{F}_{T+t}$. Then $\tilde{X}$ is an L-diffusion wrt $(\tilde{\mathcal{F}}_t)_{t\geq 0}$.

Proof: Fix $f \in C_b^2(\mathbb{R}^d)$. Consider the process

$$\tilde{M}_t^f := f(\tilde{X}_t) - f(\tilde{X}_0) - \int_0^t Lf(\tilde{X}_s)\,ds$$

$\tilde{M}^f$ is adapted to $(\tilde{\mathcal{F}}_t)$ and is integrable. For $A \in \tilde{\mathcal{F}}_s$ and $n \geq 0$ we have that

$$\mathbb{E}[(\tilde{M}_t^f - \tilde{M}_s^f)\cdot \mathbb{1}_{A\cap\{T\leq n\}}]$$
$$= \mathbb{E}[(M_{t+T}^f - M_{s+T}^f)\cdot \mathbb{1}_{A\cap\{T\leq n\}}]$$
$$= \mathbb{E}[(M_{t+T\wedge n}^f - M_{s+T\wedge n}^f)\cdot \mathbb{1}_{A\cap\{T\leq n\}}]$$
$$= 0 \quad (\text{OST}). \qquad A\cap\{T\leq n\} \in \mathcal{F}_{T\wedge n+s}.$$

Sending $n \to \infty$

$$\Rightarrow \mathbb{E}[(\tilde{M}_t^f - \tilde{M}_s^f)\cdot\mathbb{1}_A] = 0 \quad (\text{DCT})$$
$$\Rightarrow \tilde{M}^f \text{ is an } (\tilde{\mathcal{F}}_t)\text{-MG.}$$

$\square$

Lemma: Let X be an L-diffusion. Then for all $f \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ the process

$$M_t^f = f(t, X_t) - f(0, X_0) - \int_0^t (\partial/\partial s + L) f(s, X_s)\,ds$$

is a MG.

Proof: Fix $T > 0$ and consider

$$Z_n = \sup_{\substack{0\leq s\leq t\leq T,\\ t-s\leq 1/n}} |\dot{f}(s, X_t) - \dot{f}(s, X_s)| + \sup_{\substack{0\leq s\leq t\leq T,\\ t-s\leq 1/n}} |Lf(s, X_t) - Lf(t, X_t)|$$

Then Z_n is bounded and $Z_n \to 0$, $n \to \infty$ by continuity and the bounded convergence thm $\Rightarrow$ $\mathbb{E}[Z_n] \to 0$ as $n \to \infty$.

$$\begin{aligned}
M_t^f - M_s^f = &\left(f(t, X_t) - f(s, X_t) - \int_s^t \dot{f}(r, X_t)\,dr\right)\\
s<t \quad &+ \left(f(s, X_t) - f(s, X_s) - \int_s^t Lf(s, X_r)\,dr\right)\\
&+ \left(\int_s^t (\dot{f}(r, X_t) - \dot{f}(r, X_r))\,dr\right)\\
&+ \left(\int_s^t (Lf(s, X_r) - Lf(r, X_r))\,dr\right)
\end{aligned}$$

Choose $s_0 < s_1 < \dots < s_n$ with $s_0 = s$, $s_m = t$. $s_{k+1} - s_k \leq 1/n$ for each k.

- First line $= 0$ by the FTC.
- Second line has $\mathbb{E}[\dots|\mathcal{F}_s] = 0$ (X is an L-diffusion).
- For the last two lines, we have that

$$\mathbb{E}[|\mathbb{E}[M_{s_{k+1}}^f - M_{s_k}^f | \mathcal{F}_s]|] \leq (s_{k+1} - s_k)\cdot \mathbb{E}Z_n$$
$$\Rightarrow \mathbb{E}[|\mathbb{E}[M_t^f - M_s^f | \mathcal{F}_s]|] \leq (t-s)\cdot\mathbb{E}Z_n \longrightarrow 0 \quad \text{as } n\to\infty$$
$$\Longrightarrow \mathbb{E}[M_t^f | \mathcal{F}_s] = M_s^f$$

$\square$

<u>Dirichlet and Cauchy Problem</u>

Assume that a, b Lipschitz, a UPD $(a(x)\xi, \xi) \geq \varepsilon|\xi|^2$ $\forall x, \xi \in \mathbb{R}^d$. Let $D \subseteq \mathbb{R}^d$ be a bounded, open domain with smooth boundary.

We will assume the following theorem:

Theorem (Dirichlet problem) For all $f \in C(\partial D)$, $\varphi \in C(\bar{D})$, there exists a unique function $u \in C(\bar{D}) \cap C^2(D)$ such that:

$$\begin{cases} Lu + \varphi = 0 & \text{in } D\\ u = f & \text{on } \partial D.\end{cases}$$

Moreover, there exist continuous functions $m : D\times\partial D \to (0, \infty)$, $g : \{(x,y) \in D\times D : x \neq y\} \to (0, \infty)$ such that for all f, φ as above, we have that

$$u(x) = \int_D g(x,y)\varphi(y)\,dy + \int_{\partial D} f(y)\,m(x,y)\,\lambda(dy)$$

g = Green kernel, $m(x,y)\,d\lambda(y)$ harmonic measure on ∂D as seen from x.

Theorem: Suppose that $u \in C(\bar{D}) \cap C^2(D)$ satisfies

$$\begin{cases} Lu + \varphi = 0 & \text{on } D\\ u = f & \text{on } \partial D\end{cases}$$

with $f \in C(\partial D)$, $\varphi \in C(\bar{D})$. Then for any L-diffusion X starting from $x \in D$ we have that $u(x) = \mathbb{E}_x\left[\int_0^T \varphi(X_s)\,ds + f(X_T)\right]$ where $T = \inf\{t\geq 0 : X_t \notin D\}$. Moreover for all Borel sets $A \subseteq D$, $B \subseteq \partial D$. Then

$$\mathbb{E}_x\left[\int_0^T \mathbb{1}(X_s\in A)\,ds\right] = \int_A g(x,y)\,dy$$
$$\mathbb{P}_x[X_T \in B] = \int_B m(x,y)\,\lambda(dy).$$

<u>Proof:</u> Fix $n \geq 1$ and let $T_n = \inf\{t \geq 0 : X_t \notin D_n\}$ where $D_n = \{x \in D : dist(x, D^c) > 1/n\}$.
Consider $M_t = u(X_{t\wedge T_n}) - u(X_0) + \int_0^{t\wedge T_n} \varphi(X_s)ds$

There exists $\tilde{u} \in C_b^2(\mathbb{R}^d)$ with $u = \tilde{u}$ on D_n.
Then $M = \tilde{M}^{T_n}$ where:
$$\tilde{M}_t = \tilde{u}(X_t) - \tilde{u}(X_0) - \int_0^t L\tilde{u}(X_s)\,ds$$

Since X is an L-diffusion, $\tilde{M}$ is a MG.
OST $\Rightarrow$ M is a MG.
$$\Rightarrow \quad u(x) = \mathbb{E}_x\left[u(X_{t\wedge T_n}) + \int_0^{t\wedge T_n} \varphi(X_s)ds\right] \quad \circledast$$
Want to send $n \to \infty$. First will show $\mathbb{E}_x[T] < \infty$.
Take $\varphi \equiv 1$, $f \equiv 0$, let $u^{1,0}$ be the solution of the associated Dirichlet problem. Then $\circledast$ holds for $u^{1,0}$, so:
$$\mathbb{E}[T_n \wedge t] = u^{1,0}(x) - \mathbb{E}_x[u^{1,0}(X_{t\wedge T_n})]$$
Since $u^{1,0}$ is bdded $(\in C^0(\bar{D}))$, $T_n \uparrow T$ as $n \to \infty$, MCT $\Rightarrow$ $\mathbb{E}_x[T] < \infty$ $(n \to \infty, t \to \infty)$.

Now return to the general case in $\circledast$.
Have that $T_n \wedge t \uparrow T$ as $n, t \to \infty$. Since u is continuous on $\bar{D}$,
$$u(X_{t\wedge T_n}) \longrightarrow f(X_T) \text{ as } n, t \to \infty.$$
Since u is bounded on $\bar{D}$ ($\bar{D}$ compact, u continuous), bounded convergence theorem $\Rightarrow$ $\mathbb{E}_x[u(X_{t\wedge T_n})] \longrightarrow \mathbb{E}_x[f(X_T)]$ as $t, n \to \infty$. Moreover,
$$\mathbb{E}_x\left[\int_0^T |\varphi(X_s)|ds\right] \leq \|\varphi\|_\infty \cdot \mathbb{E}_x[T] < \infty.$$
DCT $\Rightarrow$ $\mathbb{E}_x\left[\int_0^{t\wedge T_n} \varphi(X_s)ds\right] \to \mathbb{E}_x\left[\int_0^T \varphi(X_s)ds\right]$
Thus, $\quad u(x) = \mathbb{E}_x\left[f(X_T) + \int_0^T \varphi(X_s)ds\right]$.

Final assertions follow by taking limits as $\varphi_n \longrightarrow \mathbb{1}_A$, $f \equiv 0$ and $f_n \to \mathbb{1}_B$, $\varphi \equiv 0$ $\square$.

<u>Cauchy Problem:</u>

<u>Theorem:</u> For each $f \in C_b^2$, there exists a unique solution $u \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ such that:
$$\begin{cases} \partial u/\partial t = Lu & \text{on } \mathbb{R}_+ \times \mathbb{R}^d \\ u(0, \cdot) = f & \text{on } \mathbb{R}^d \end{cases}$$

Moreover, there exists a continuous function $p : (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to (0,\infty)$ such that
$$u(t,x) = \int_{\mathbb{R}^d} \underbrace{p(t,x,y)}_{\text{"heat kernel"}} f(y)\,dy \quad \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d.$$

<u>Theorem:</u> Assume that $f \in C_b^2(\mathbb{R}^d)$. Let u satisfy $\begin{cases} \partial u/\partial t = Lu & \text{on } \mathbb{R}_+ \times \mathbb{R}^d \\ u(0,\cdot) = f & \text{on } \mathbb{R}^d. \end{cases}$
Then for any L-diffusion X starting from x, for all $t \in \mathbb{R}_+$, $0 \leq s \leq t$ we have that
$$\mathbb{E}_x[f(X_t) | \mathcal{F}_s] = u(t-s, X_s) \quad \text{a.s.}$$

In particular, $\mathbb{E}_x[f(X_t)] = u(t,x) = \int_{\mathbb{R}^d} p(t,x,y)f(y)dy$

Finally, under $\mathbb{P}_x$, the finite dimensional distributions of X are given by:
$$\mathbb{P}_x[X_{t_1} \in dx_1, \dots, X_{t_n} \in dx_n]$$
$$= p(t_1, x_0, x_1) \times \dots \times p(t_n - t_{n-1}, x_{n-1}, x_n)dx_1 \dots dx_n$$
$0 < t_1 < t_2 \dots < t_n < \infty$, $x_1, \dots, x_n \in \mathbb{R}^d$, $x_0 = x$.

<u>Proof:</u> Fix $t \in (0,\infty)$. Consider $g(s,x) = u(t-s,x)$ $s \leq t$, $x \in \mathbb{R}^d$. Note that $(\partial/\partial s + L)g(s,x) = -\dot{u}(t-s,x) + Lu(t-s,x)$
$= 0$.
$$\Rightarrow \quad M_s^g = g(s,X_s) - g(0,X_0) - \int_0^t \cancelto{0}{\left(\frac{\partial}{\partial s} + L\right)} g(s,X_s)ds$$
$$= g(s,X_t) - g(0,X_0) \text{ is a MG for}$$
$s \in [0,t)$ by extending g to $\tilde{g} \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ appropriately. Hence, $\forall$ $0 < s < t' < t$:
$\mathbb{E}[M_{t'}^g | \mathcal{F}_s] = M_s^g$ a.s. $\Rightarrow$ $\mathbb{E}_x[M_{t'}^g] = \mathbb{E}_x[M_0^g]$ $\forall t' \in [0,t)$
$\Rightarrow \mathbb{E}_x[u(t-t', X_{t'})] = u(t,x)$. Now, as $t' \uparrow t$, by continuity, $u(t-t', X_{t'}) \to f(X_t) \Rightarrow$ (Bdd conv; $u \in C_b^2$): $\mathbb{E}_x[f(X_t)] = u(t,x)$.
Second part of the theorem. Set
$$P_t f(x) = \int_{\mathbb{R}^d} p(t,x,y)f(y)dy = u(t,x)$$

Uniqueness of solutions to Cauchy problem
$\Rightarrow P_s(P_t f) = P_{s+t} f$
Claim (by induction):
$$\mathbb{E}\left[\prod_{i=1}^n f_i(X_t)\right] = \int_{(\mathbb{R}^d)^n} p(t_1, x_0, x_1)f(x_1) \dots$$
$$p(t_n - t_{n-1}, x_{n-1}, x_n)f(x_n)$$
$$dx_1 \dots dx_n.$$

For induction, we use that:
$$\mathbb{E}_{x_0}\left[\prod_{i=1}^n f_{t_i}(X_{t_i}) \Big| \mathcal{F}_{t_{n-1}}\right]$$
$$= \prod_{i=1}^{n-1} f_{t_i}(X_{t_i})\, \mathbb{E}[f_{t_n}(X_{t_n}) | \mathcal{F}_{t_{n-1}}]$$
$$= \prod_{i=1}^{n-1} f_{t_i}(X_{t_i}) P_{t_n - t_{n-1}} f(X_{t_{n-1}})$$

Now apply the case $n-1$. $\square$