Imperial College London

COURSEWORK 1

IMPERIAL COLLEGE LONDON

DEPARTMENT OF MATHEMATICS

MATH60028
Probability Theory

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Problems

Question 1

Part (a)

Let H and Y are independent and identical distributed random variables, where

$$F_H(t) = F_Y(t) = \int_{\{s\le t\}} \mathbb{1}_{[0,1]}(s)ds = \begin{cases} 1, & t \ge 1 \\ t, & t \in (0,1) \\ 0, & t \le 0 \end{cases}$$
$$= z \cdot \mathbb{1}_{[0,1)}(z) + \mathbb{1}_{[1,\infty)}(z), \quad z \in \mathbb{R} \tag{1}$$

is the cumulative distribution function of a uniform random variable on $[0,1]$. Compute the distribution of $Z = H + Y$.

Part (b)

Suppose that X, Y are independent random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and furthermore that Y is uniformly distributed on $[0,1]$. Recall that the fractional part

$$\{X+Y\} = X + Y - \lfloor X+Y \rfloor \in [0,1) \tag{2}$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x \in \mathbb{R}$. By (8), to compute the distribution of $\{X+Y\}$.

Question 2

Let $\zeta \sim \mathcal{N}(m_1, \sigma_1^2)$ and $\eta \sim \mathcal{N}(m_2, \sigma_2^2)$ be independent normally distributed random variables with densities

$$f_\zeta(s) = \frac{1}{\sigma_1} \cdot \phi\left(\frac{(s-m_1)}{\sigma_1}\right) \text{and } f_\eta(s) = \frac{1}{\sigma_2} \cdot \phi\left(\frac{(s-m_2)}{\sigma_2}\right), \quad \sigma \in \mathbb{R}$$

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \quad x \in \mathbb{R}.$$

Compute the law of $\zeta + \eta$.

Question 3

Let H be an integrable non-negative real-valued random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with distribution function $F(x)$. Furthermore, let

$$G(x) = \begin{cases} \frac{1}{\mathbb{E}[H]} \int_0^x 1 - F(s)ds, & x \in [0,\infty) \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

Show that G is a distribution function.

Question 4

Let ξ be a non-negative rando variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, its expectation is defined as:

$$\mathbb{E}[\xi] = \sup_{n\in\mathbb{N}} \mathbb{E}[\xi_n], \quad \xi_n \uparrow \xi \tag{4}$$

where the $(\xi_n)_{n\in\mathbb{N}}$ are an increasing sequence of simple functions. Show that the above definition is independent of the choice of $(\xi_n)_{n\in\mathbb{N}}$, and so is well-defined.

Solutions

Question 1

Part (a)

From lectures, since H and Y are independent and identical distributed random variables, their joint density factorises as follows:

$$F_{H,Y}(x,y) := \mathbb{P}(H \leq x, Y \leq y) = \mathbb{P}(H \leq x) \cdot \mathbb{P}(Y \leq y) \tag{5}$$
$$= F_H(x) \cdot F_Y(y), \quad x, y \in \mathbb{R}$$

where

$$F_H(t) = F_Y(t) = \int_{\{s \leq t\}} \mathbb{1}_{[0,1]}(s) ds = \begin{cases} 1, & t \geq 1 \\ t, & t \in (0,1) \\ 0, & t \leq 0 \end{cases}$$
$$= z \cdot \mathbb{1}_{[0,1)}(z) + \mathbb{1}_{[1,\infty)}(z), \quad z \in \mathbb{R} \tag{6}$$

is the cumulative distribution function of a uniform random variable on $[0,1]$. Now, from page 28 of the lecture notes, the distribution of

$$Z = H + Y$$

can be computed as follows:

$$F_{H+Y}(z) = \mathbb{E}[\mathbb{1}_{\{H+Y \leq z\}}] = \int_\Omega \mathbb{1}_{\{H+Y \leq z\}}(\omega) d\mathbb{P}(\omega) = \int_\mathbb{R} \int_\mathbb{R} \mathbb{1}_{\{x+y \leq z\}}(\omega) dF_H(x) dF_Y(y)$$
$$= \int_\mathbb{R} F_H(z-y) dF_Y(y) = \int_\mathbb{R} F_H(z-s) \mathbb{1}_{[0,1]}(s) ds = \int_0^1 F_H(z-s) ds$$
$$= \int_0^1 (z-s) \cdot \mathbb{1}_{[0,1)}(z-s) + \mathbb{1}_{[1,\infty)}(z-s) ds$$
$$= \int_0^1 (z-s) \cdot \mathbb{1}_{(z-1,z]}(s) + \mathbb{1}_{(-\infty,z-1]}(s) ds$$
$$= \int_{(-\infty,z]} s \cdot \mathbb{1}_{(0,1]}(s) + (2-s) \cdot \mathbb{1}_{(1,2]}(s) ds, \quad z \in \mathbb{R}$$
$$= \begin{cases} 1, & z \geq 2 \\ 2z - \frac{1}{2}z^2 - 1, & z \in (1,2) \\ \frac{1}{2}z^2, & z \in (0,1] \\ 0, & z \leq 0 \end{cases} \tag{7}$$

Thus, the density of $Z = H + Y$ with respect to the Lebesgue measure is:

$$f_Z(z) = z \cdot \mathbb{1}_{(0,1]}(z) + (2-z) \cdot \mathbb{1}_{(1,2]}(z), \quad z \in \mathbb{R}$$

Part (b)

Suppose that X, Y are independent random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and furthermore that Y is uniformly distributed on $[0,1]$. By definition, the fractional part

$$\{X+Y\} = X + Y - \lfloor X+Y \rfloor \in [0,1) \tag{8}$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x \in \mathbb{R}$. By (8), to compute the density, it suffices to restrict one's attention to $z \in [0,1)$ and compute:

$$F_{\{X+Y\}}(z) = \mathbb{P}(\{X+Y\} \le z) = \begin{cases} 1, & z \in [1,\infty) \\ g(z), & z \in [0,1) \\ 0, & z \in (-\infty, 0) \end{cases} \tag{9}$$

for some $g : [0,1) \to \mathbb{R}_{\ge 0}$ to be determined.

Now, for $z \in [0,1)$,

$$\begin{aligned} g(z) &= \mathbb{P}(\{X+Y \in (-\infty, z])\}) \\ &= \sum_{n=-\infty}^{\infty} \mathbb{P}(X+Y \in [n, n+z]) \end{aligned}$$

by definition of (8). Additionally, from the independence of X and Y, the distribution of their sum is as follows:

$$\mathbb{P}(X+Y \le z) = \int_{\mathbb{R}} F_Y(z-x)dF_X(x), \quad z \in \mathbb{R} \tag{10}$$

One notices that (10) is continuous in z. this follows from the continuity of F_Y since Y is uniformly distributed and is absolutely continuous with respect to the Lebesgue measure. Thus, we obtain

$$\begin{aligned} &\mathbb{P}(X+Y \in [n, n+z]) = \mathbb{P}(X+Y \in (n, n+z]) \\ &= \mathbb{P}(X+Y \in (-\infty, n+z]) - \mathbb{P}(X+Y \in (-\infty, n]) \end{aligned}$$

Now,

$$\begin{aligned} g(z) &= \sum_{n=-\infty}^{\infty} \mathbb{P}(X+Y \in (-\infty, n+z]) - \mathbb{P}(X+Y \in (-\infty, n]) \\ &= \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}} F_Y(n+z-x) - F_Y(n-x)dF_X(x) \end{aligned}$$

Note that the double sums in what is to follow are defined as

$$\sum_{n=-\infty}^{\infty} := \lim_{N\to\infty} \sum_{n=-N}^{N} \tag{11}$$

and are shown to converge. I now claim that $g(z) = z$. To see this, first note that Y is uniformly distributed on $[0,1]$ giving:

$$F_Y(z) = z \cdot \mathbb{1}_{[0,1)}(z) + \mathbb{1}_{[1,\infty)}(z), \quad z \in \mathbb{R}$$

Thus, we can re-express $g(z)$ as:

$$g(z) = \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}} \Big((n+z-x) \cdot \mathbb{1}_{(n+z-1,n+z]}(x) + \mathbb{1}_{(-\infty,n+z-1]}(x) - (n-x) \cdot \mathbb{1}_{(n-1,n]}(x) - \mathbb{1}_{(-\infty,n-1]}(x) \Big) dF_X(x)$$

$$= \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}} \Big(z \cdot \mathbb{1}_{(n+z-1,n+z]}(x) + \mathbb{1}_{(n-1,n+z-1]}(x) \Big) dF_X(x) + \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}} \Big((n-x) \cdot \mathbb{1}_{(n+z-1,n+z]}(x) - (n-x) \cdot \mathbb{1}_{(n-1,n]}(x) \Big) dF_X(x) \tag{12}$$

Note that the decomposition in equation (12) is valid since g(z) and the expression on the third line are absolutely convergent series (with non-negative summands) implying by the algebra of limits that the last expression converges in the sense of (11). Incidentally, the last term in (12) is a limit of sum of uniformly bounded functions on $\mathbb{R}$, hence the integral and the summation can be exchanged, making the limit well-defined. Now, exploiting the fact that $z \in [0,1)$ and that

$$\bigsqcup_{n=-\infty}^{\infty} (n+z-1, n+z] = \bigsqcup_{n=-\infty}^{\infty} (n-1, n+z-1] = \mathbb{R}$$

we obtain:

$$g(z) = z + A + \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}} (n-x) \cdot \Big(\mathbb{1}_{(n,n+z]}(x) - \mathbb{1}_{(n-1,n-1+z]}(x) \Big) dF_X(x)$$

where

$$A = \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}} \mathbb{1}_{(n-1,n+z-1]}(x) dF_X(x) = \int_{\mathbb{R}} \sum_{n=-\infty}^{\infty} \mathbb{1}_{(n-1,n+z-1]}(x) dF_X(x)$$

$$= z + A + \lim_{N\to\infty} \sum_{n=-N}^{N} \int_{\mathbb{R}} (n-x) \cdot \Big(\mathbb{1}_{(n,n+z]}(x) - \mathbb{1}_{(n-1,n-1+z]}(x) \Big) dF_X(x)$$

by dominated convergence. Now, let us consider the series with terms:

$$\sum_{n=-N}^{N} (n-x) \cdot \Big(\mathbb{1}_{(n,n+z]}(x) - \mathbb{1}_{(n-1,n-1+z]}(x) \Big)$$

$$= \sum_{n=-N}^{N} (n-x) \cdot \mathbb{1}_{(n,n+z]}(x) - \sum_{n=-N}^{N} (n-x) \cdot \mathbb{1}_{(n-1,n-1+z]}(x)$$

$$= \sum_{n=-N}^{N} (n-x) \cdot \mathbb{1}_{(n,n+z]}(x) - \sum_{n=-(N+1)}^{N-1} (n+1-x) \cdot \mathbb{1}_{(n,n+z]}(x)$$

$$= (N - x) \cdot \mathbb{1}_{[N,N+z)}(x) - (-N - 1 - x) \cdot \mathbb{1}_{(-N-1,-N-1+z]}(x) - \sum_{n=-(N+1)}^{N-1} \mathbb{1}_{(n,n+z]}(x)$$

Now, $g(z)$ becomes:

$$g(z) = z + A + \lim_{N\to\infty} (\alpha_N - \alpha_{-N-1}) - \int_{\mathbb{R}} \sum_{n=-\infty}^{\infty} \mathbb{1}_{(n,n+z]}(x) dF_X(x)$$

where

$$\alpha_N = \int_{\mathbb{R}} (N - x) \cdot \mathbb{1}_{(N,N+z]}(x) dF_X(x), \quad N \in \mathbb{Z}$$

One can readily obtain estimates on α_N giving:

$$|\alpha_N| \leq \int_{\mathbb{R}} |(N - x)| \cdot \mathbb{1}_{(N,N+z]}(x) dF_X(x) \leq z \cdot \mathbb{P}(X \in (N, N + z])$$

Now, notice that

$$\sum_{n=-\infty}^{\infty} \mathbb{P}(X \in (n, n + z]) = \mathbb{P}\left(X \in \bigsqcup_{n=-\infty}^{\infty} (n, n + z]\right) \leq 1$$

since $z \in [0, 1)$. Thus, as $|N| \to \infty$, one has that $\mathbb{P}(X \in (n, n + z]) \to 0$, bounded above by the tail of a convergent series. Hence, $\lim_{|N|\to\infty} \alpha_N = 0$ finally yielding:

$$g(z) = z + A + 0 - \int_{\mathbb{R}} \sum_{n=-\infty}^{\infty} \mathbb{1}_{(n,n+z]}(x) dF_X(x)$$

$$= z + A + 0 - \int_{\mathbb{R}} \sum_{n=-\infty}^{\infty} \mathbb{1}_{(n-1,n-1+z]}(x) dF_X(x) = z + A - A = z$$

by relabelling the absolutely convergent sum in the above integral. This means that, in light of the above and (9), $\{X + Y\} \sim U[0, 1]$, i.e. is uniformly distributed on $[0, 1]$.

Question 2

Let $\zeta \sim \mathcal{N}(m_1, \sigma_1^2)$ and $\eta \sim \mathcal{N}(m_2, \sigma_2^2)$ be independent normally distributed random variables with densities

$$f_\zeta(s) = \frac{1}{\sigma_1} \cdot \phi\left(\frac{(s-m_1)}{\sigma_1}\right) \text{and } f_\eta(s) = \frac{1}{\sigma_2} \cdot \phi\left(\frac{(s-m_2)}{\sigma_2}\right), \quad \sigma \in \mathbb{R}$$

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \quad x \in \mathbb{R}$$

From lectures, the density of $\zeta + \eta$ is

$$f_{\zeta+\eta}(s) = \int_{\mathbb{R}} f_\zeta(s-t) f_\eta(t) dt$$

$$= \frac{1}{2\pi\sigma_1\sigma_2} \int_{\mathbb{R}} \exp\left(\frac{(s-t-m_1)^2}{2\sigma_1^2}\right) \cdot \exp\left(\frac{(t-m_2)^2}{2\sigma_2^2}\right) dt$$

$$= \frac{1}{2\pi\sigma_1\sigma_2} \cdot \exp\left[-\frac{(s-m_1)^2}{2\sigma_1^2} - \frac{m_2^2}{2\sigma_2^2}\right] \cdot \int_{\mathbb{R}} \exp\left[-\frac{1}{2}\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right) t^2 + \left(\frac{s-m_1}{\sigma_1^2} + \frac{m_2}{\sigma_2^2}\right) t\right] dt$$

$$= \frac{1}{2\pi\sigma_1\sigma_2} \cdot \exp\left[-\frac{(s-m_1)^2}{2\sigma_1^2} - \frac{m_2^2}{\sigma_2^2}\right] \cdot \int_{\mathbb{R}} \exp\left[-\frac{1}{2}\left(\frac{1}{\sigma_1^2} + \frac{1}{2\sigma_2^2}\right) \cdot \left(t - \frac{(s-m_1)\sigma_2^2 + m_2\sigma_1^2}{\sigma_1^2 + \sigma_2^2}\right)^2\right] dt$$

$$\times \exp\left[\frac{(s-m_1)^2\sigma_2^2}{2\sigma_1^2(\sigma_1^2+\sigma_2^2)} + \frac{m_2\sigma_1^2}{2\sigma_2^2(\sigma_1^2\sigma_2^2)} + \frac{(s-m_1)m_2}{(\sigma_1^2+\sigma_2^2)}\right]$$

$$= \frac{1}{\sqrt{2\pi} \cdot \sqrt{\sigma_1^2 + \sigma_2^2}} \cdot \exp\left[-\frac{(s-m_1)^2}{2(\sigma_1^2+\sigma_2^2)} - \frac{m_2^2}{2(\sigma_1^2+\sigma_2^2)} + \frac{m_2(s-m_1)}{\sigma_1^2+\sigma_2^2}\right]$$

$$= \frac{1}{\sqrt{2\pi} \cdot \sqrt{\sigma_1^2 + \sigma_2^2}} \cdot \exp\left[-\frac{(s-m_1-m_2)^2}{2(\sigma_1^2+\sigma_2^2)}\right], \quad s \in \mathbb{R}$$

Thus, $\zeta + \eta$ has the density of a

$$\mathcal{N}(m_1 + m_2, \sigma_1^2 + \sigma_2^2)$$

random variable as required.

Question 3

Consider H an integrable non-negative real-valued random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with distribution function $F(x)$. Furthermore, let

$$G(x) = \begin{cases} \frac{1}{\mathbb{E}[H]} \int_0^x 1 - F(s) ds, & x \in [0, \infty) \\ 0, & \text{otherwise} \end{cases} \tag{13}$$

We now check that G is indeed a distribution function. First, we have

$$\lim_{x \to -\infty} G(H) = 0$$

since $G(x) = 0$ for $x < 0$. Now fix $0 \leq x \leq y$. Since, $\mathbb{1}_{[0,x]} \leq \mathbb{1}_{[0,y]}$ and $1 - F(s) \geq 0$, for all $S \in \mathbb{R}$ as F is a distribution function one computes

$$G(x) = \frac{1}{\mathbb{E}[H]} \int_{\mathbb{R}} \mathbb{1}_{[0,x]}(s) \cdot (1 - F(s)) ds \leq \frac{1}{\mathbb{E}[H]} \int_{\mathbb{R}} \mathbb{1}_{[0,y]}(s) \cdot (1 - F(s)) ds = G(y)$$

by the monotonicity of the Lebesgue integral. The other cases for x, y are easily dealt with by the non-negativity of G. Hence, the limit

$$\lim_{x \to \infty} G(H)$$

exists and we now compute it. By monotone convergence,

$$\lim_{x \to \infty} G(H) = \frac{1}{\mathbb{E}[H]} \int_0^\infty (1 - F(s)) ds = \frac{1}{\mathbb{E}[H]} \int_0^\infty \mathbb{P}(H > s) ds$$

$$= \frac{1}{\mathbb{E}[H]} \int_0^\infty \int_\Omega \mathbb{1}_{\{H > s\}} ds = \frac{1}{\mathbb{E}[H]} \int_{\mathbb{R}} \int_\Omega \mathbb{1}_{\{H > s\}}(\omega) d\mathbb{P}(\omega) ds$$

Now, by Tonelli's Theorem -since the integrand is non-negative and jointly measurable- we exchange the order of integration to obtain

$$\frac{1}{\mathbb{E}[H]} \int_0^\infty \mathbb{P}(H > s) ds = \frac{1}{\mathbb{E}[H]} \int_0^\infty \int_\Omega \mathbb{1}_{\{H > s\}}(\omega) d\mathbb{P}(\omega) ds$$

$$= \frac{1}{\mathbb{E}[H]} \int_\Omega \int_{\mathbb{R}} \mathbb{1}_{\{H(\omega) > s\}}(s) \cdot \mathbb{1}_{\{0 \leq s < \infty\}}(s) ds d\mathbb{P}(\omega)$$

$$= \frac{1}{\mathbb{E}[H]} \int_\Omega \int_0^{H(\omega)} ds d\mathbb{P}(\omega) = \frac{1}{\mathbb{E}[H]} \int_\Omega H(\omega) d\mathbb{P}(\omega) = \frac{\mathbb{E}[H]}{\mathbb{E}[H]} = 1$$

We now show that $G(x)$ is continuous. First, consider $x \in (0, \infty)$ and a sequence $x_n \to x, \quad n \to \infty$. Without loss of generality, assume that $x_n > 0$ for all $n \in \mathbb{N}$. This means that

$$f_n = \mathbb{1}_{[\min\{x_n, x\}, \max\{x_n, x\}]}(s) \cdot \frac{(1 - F(s))}{\mathbb{E}[H]} \to 0$$

almost everywhere as $n \to \infty$. Now,

$$|f_n| \leq \frac{(1-F(s))}{|\mathbb{E}[H]|}$$

which is integrable by the above argument. Hence, by Lebesgue's dominated convergence theorem,

$$\begin{aligned}
|G(x_n) - G(x)| &= \left| \frac{1}{\mathbb{E}[H]} \int_0^x (1-F(s))ds - \frac{1}{\mathbb{E}[H]} \int_0^{x_n} (1-F(s))ds \right| \\
&= \left| \int_{\mathbb{R}} f_n(s)ds \right| \to 0, \quad n \to \infty
\end{aligned}$$

The same argument yields that from above

$$G(x_n) \to G(0) = 0, \quad x_n \downarrow 0$$

as $n \to \infty$. Finally note that for $x \in (-\infty, 0]$, $G(x) = 0$, which is clearly continuous. Thus, $G(x)$ is continuous on $\mathbb{R}$, which shows that it is a distribution function. In fact, we have shown that it is a distribution function of a **continuous** random variable. Integrability of G is equivalent to the condition

$$\int_{\mathbb{R}} |x| dG(x) = \int_{[0,\infty)} |x| dG(x) < \infty \tag{14}$$

Note that for Borel measurable sets $A \subseteq \mathbb{R}$,

$$\int_{\mathbb{R}} \mathbb{1}_A dG(x) = G(A) = \int_{\mathbb{R}} \mathbb{1}_A \cdot \frac{1}{\mathbb{E}[H]} \cdot (1-F(s))ds$$

this equality extends to simple functions by linearity of the integrals, and can be extended once more to integrable (Borel-measurable) $g : \mathbb{R} \to \mathbb{R}$ through an approximation by simple functions and an application of Lebesgue's dominated convergence theorem. Thus, in our case we obtain that $|\cdot| \mathbb{1}_{[0,\infty)}(\cdot)$ is integrable with respect to dG if and only if

$$|s| \cdot \mathbb{1}_{[0,\infty)}(s) \cdot \frac{1}{\mathbb{E}[H]} \cdot (1-F(s))$$

is integrable with respect to the Lebesgue measure denoted by $d\lambda(s)$, or simply ds. Thus, (14) holds iff

$$\int_{\mathbb{R}} \frac{1}{\mathbb{E}[H]} \cdot |s| \cdot \mathbb{1}_{[0,\infty)}(s) \cdot (1-F(s))ds < \infty \tag{15}$$

Now, we investigate (15):

$$\begin{aligned}
\int_{\mathbb{R}} \frac{1}{\mathbb{E}[H]} \cdot |s| \cdot \mathbb{1}_{[0,\infty)}(s) \cdot (1-F(s))ds &= \int_{\mathbb{R}} \frac{1}{\mathbb{E}[H]} \cdot |s| \cdot \mathbb{1}_{[0,\infty)}(s) \mathbb{P}(H > s)ds \\
&= \int_{\mathbb{R}} \int_{\Omega} \frac{1}{\mathbb{E}[H]} \cdot |s| \cdot \mathbb{1}_{[0,\infty)}(s) \cdot \mathbb{1}_{\{H>s\}}(\omega) d\mathbb{P}(\omega) ds
\end{aligned}$$

now since the integrand is non-negative, by Tonelli's Theorem we exchange the order of integration to obtain

$$= \int_\Omega \int_{\mathbb{R}} \frac{1}{\mathbb{E}[H]} \cdot |s| \cdot \mathbb{1}_{[0,\infty)}(s) \cdot \mathbb{1}_{\{H(\omega)>s\}}(s) ds d\mathbb{P}(\omega)$$

$$= \frac{1}{\mathbb{E}[H]} \int_\Omega \int_{\mathbb{R}} |s| \cdot \mathbb{1}_{\{H(\omega)>s\}}(s) \cdot \mathbb{1}_{[0,\infty)}(s) ds d\mathbb{P}(\omega)$$

$$= \frac{1}{\mathbb{E}[H]} \int_\Omega \int_0^{H(\omega)} s ds d\mathbb{P}(\omega) = \frac{1}{2 \cdot \mathbb{E}[H]} \int_\Omega |H(\omega)|^2 d\mathbb{P}(\omega) = \frac{1}{2\mathbb{E}[\mathbb{H}]} \cdot \mathbb{E}[H^2]$$

Thus, (14) holds iff

$$\mathbb{E}[H^2] < \infty$$

that is iff H has a finite second moment/variance.

Question 4

For the non-negative random variable ξ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, its expectation is defined as:

$$\mathbb{E}[\xi] = \sup_{n\in\mathbb{N}} \mathbb{E}[\xi_n], \quad \xi_n \uparrow \xi \tag{16}$$

where the $(\xi_n)_{n\in\mathbb{N}}$ are an increasing sequence of simple functions. One can take them to be the simple functions $\xi_n \uparrow \xi$ as constructed in the lecture notes. Of course this definition can depend on our choice of ξ_n. We show that this is indeed **not** the case below.

Now, fix an arbitrary simple function $s \leq \xi$. Since $\xi_n \uparrow \xi$, and $\Omega = \{s \leq \xi\} = \{\omega \in \Omega | s(\omega) \leq \xi(\omega)\}$, it follows that for all $\epsilon > 0$:

$$\Omega = \bigcup_{m=1}^{\infty} \{\omega \in \Omega | s(\omega) - \xi_m(\omega) < \epsilon\} = \bigcup_{m=1}^{\infty} B_{m,\epsilon}$$

since s is a finite function. Also note that the $B_{m,\epsilon}$ form an increasing sequence as ξ_m is an increasing sequence of functions. Now, fix an $m \in \mathbb{N}$ and notice

$$s \cdot \mathbb{1}_{B_{m,\epsilon}} \leq (\xi_m + \epsilon) \cdot \mathbb{1}_{B_{m,\epsilon}} \leq \xi_m + \epsilon \tag{17}$$

by the non-negativity of the ξ_m. By virtue of the fact that s is simple, choose a representation

$$s = \sum_k a_k \mathbb{1}_{A_k}, \quad a_k \in \mathbb{R}$$

where k ranges over a finite set and the A_k are $\mathcal{F}$ measurable. Taking expectations (note there is no real ambiguity with the definition of expectation for simple functions) of (17) one obtains:

$$\mathbb{E}[s \cdot \mathbb{1}_{B_{m,\epsilon}}] = \sum_k a_k \cdot \mathbb{P}(A_k \cap B_{m,\epsilon}) \leq \mathbb{E}[\xi_m] + \epsilon \leq \sup_{m\in\mathbb{N}} \mathbb{E}[\xi_m] + \epsilon$$

Now, using the continuity of $\mathbb{P}$,

$$A_k \cap B_{m,\epsilon} \uparrow A_k \implies \mathbb{P}(A_k \cap B_{m,\epsilon}) \uparrow \mathbb{P}(A_k), \quad m \to \infty$$

Thus, taking $m \to \infty$ yields that for all $\epsilon > 0$:

$$\mathbb{E}[s] = \sum_k a_k \cdot \mathbb{P}(A_k) \leq \sup_{m\in\mathbb{N}} \mathbb{E}[\xi_m] + \epsilon$$

Thus,

$$\mathbb{E}[s] \leq \sup_{m\in\mathbb{N}} \mathbb{E}[\xi_m]$$

and taking suprema over $s \leq \xi$ simple yields:

$$\sup_{s\leq\xi} \mathbb{E}[s] \leq \sup_{m\in\mathbb{N}} \mathbb{E}[\xi_m]$$

Now the reverse inequality can easily be obtained since the sequence ξ_n of simple functions satisfies $\xi_n \leq \xi$. Hence,

$$\sup_{m \in \mathbb{N}} \mathbb{E}[\xi_m] \leq \sup_{s \leq \xi} \mathbb{E}[s]$$

to finally give

$$\sup_{m \in \mathbb{N}} \mathbb{E}[\xi_m] = \sup_{s \leq \xi} \mathbb{E}[s]$$

as desired. Note we have just proved that the definition of expectation does not depend on the choice of ξ_n, thus making the definition given at the beginning well-defined.