Infinite-dimensional Bayesian inference for time evolution PDEs

I. Introd

I. 1. Non-linear dynamics

$\Omega = (0,1]^d$, $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{(\partial x_i)^2}$, $L^2(\Omega)$, $H^\alpha(\Omega)$

$L^2_0 = L^2 \cap \{h : \int_\Omega h\,dx = 0\} \cap (\{\nabla \cdot h = 0\}$ if $h : \Omega \to \mathbb{R}^2$.)

$P : L^2 \to L^2_0$ (projection operator).

Initial condition $\theta = u(0, \cdot)$
Consider $(u_\theta(t,x) : t \in [0,T], x \in \Omega)$ solves a PDE

(Reaction-diffusion model)
$$\frac{\partial}{\partial t} u - \Delta u - \underset{\text{non-linear}}{f \circ u} = 0 \text{ on } (0,T] \times \Omega$$
also $f : \mathbb{R} \to \mathbb{R}$.

Look at $f \in C^\infty_c(\mathbb{R})$
or in $d = 2$:

$$\frac{\partial}{\partial t} u - \underset{\text{projection}}{\overset{\text{"viscosity"}}{\nu P \Delta u}} + \overset{\text{advection term, bilinear}}{B(u,u)} = 0 \text{ on } [0,T] \times \Omega$$

where $B(u,u) = P(u \cdot \nabla)u$, $[(u \cdot \nabla) u]_i = \sum_{j=1}^{2} u_j \frac{\partial u_i}{\partial x_j}$.

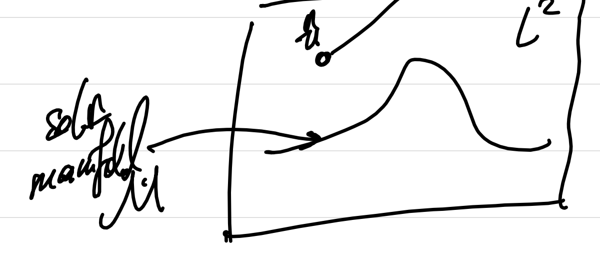
Typically $\theta \in H^1 \cap L^2_0 = H^1_0$.

<u>Proposition:</u> Let $\theta \in H^1_0$, $T > 0$, then $\exists!$ u_θ to these PDEs ($f \in C^\infty_c(\mathbb{R})$, $d = 2$).

I.2. Discrete observations

Consider $(Y_i, t_i, X_i)_{i=1}^N$, N sample size follow a regression $Y_i = u_\theta(t_i, X_i) + \varepsilon_i$ where $\varepsilon_i \overset{iid}{\sim} N(0,1)$ and independently "prob. numerics" $(t_i, X_i) \overset{iid}{\sim} U([0,T] \times \Omega)$, (Diaconis 1988).

<u>Notation:</u> $Z^{(N)} := (Y_i, t_i, X_i)_{i=1}^N$
Data assimilation: LSZ (2015).
Goal: infer $u_\theta(t)$ from $Z^{(N)}$, where (w.p. $\theta \in M$).
$t = t_{(N)} = \max_{i \le N} t_i \longrightarrow$ <u>filtering</u>. estimator



The joint distribution of $Z^{(N)}$ has prob. density $p^N_\theta = \prod_{i=1}^N p_\theta(y_i, t_i, x_i)$ where

$p_\theta(y,t,x) \propto \exp(-\frac{1}{2}|y - u_\theta(t,x)|^2)$. The log-likelihood is $\ell_N(\theta) = -\frac{1}{2} \sum_{i=1}^N (Y_i - u_\theta(t_i, X_i))^2$,
$\theta \in M \subseteq H^1_0$

<u>Note:</u> $-\ell_N(\theta)$ is <u>not</u> convex, so optimisation.

I.3. Gaussian process priors.

For θ we consider "prior" Gaussian random fields $(\Theta(x) : x \in \Omega)$ over L^2_0 of the form $\Theta \sim \Pi_\gamma \sim N(0, \rho^2(-\Delta)^{-\gamma})$, $\rho > 0$, $\gamma > 1 + d/2$, so $\theta \in H^1$ a.s. (in fact in $H^\beta_0\ \forall \beta \in (0, \gamma - d/2)$).
$\hookrightarrow$ smoothness modulator

If (e_j, λ_j) are s.t. $\Delta e_j = -\lambda_j e_j$ then
$$\Theta = \rho \Theta' \overset{a.s.}{\underset{L^2}{=}} \rho \sum_{j=1}^\infty \lambda_j^{-\gamma} g_j e_j, \quad g_j \overset{iid}{\sim} N(0,1).$$
Note we obtain a prior $\Pi \circ u_\theta^{-1}$ in $C([0,T], L^2(\Omega))$.

I4. Posterior measures.

Suppose $(\theta, t, x) \longmapsto u_\theta(t,x)$ is jointly measurable for some σ-field over $M \times [0,T] \times \Omega$, then $p^N_\theta(y,t,x)$ is jointly measurable and if we define a new density dQ on $M \times \mathbb{R} \times [0,T] \times \Omega$

$$dQ(\theta, y, t, x) = p_\theta(y, t, x)\,dy\,dt\,dx\,d\Pi_\gamma(\theta)$$

to see that

(*) $Z^{(N)} | \theta \sim P^N_\theta$ and the posterior dist. is

(*) $$\Theta |_{(Y_i, t_i, X_i)_{i=1}^N} \sim \Pi(\theta | Z^{(N)}) \sim \frac{\prod_{i=1}^N p_\theta(Y_i, t_i, X_i)\,d\Pi_\gamma(\theta)}{\int_M \prod_{i=1}^N p_\theta(Y_i, t_i, X_i)\,d\Pi_\gamma(\theta)}$$
$$\sim e^{\ell_N(\theta)}\,d\Pi(\theta).$$

(*) $$u_\theta | Z^{(N)} \sim \underbrace{\text{Law}(u_\theta : \theta \sim \Pi(\cdot | Z^{(N)}))}_{(\hat{\Pi}_t : t \ge 0)}.$$

$\hat{\Pi}_{t_N}$: filtering distribution.
$\hookrightarrow \max_{i \le N} t_i$

I.5 Posterior computation

Note the $\overset{\text{max. aposteriori}}{\text{MAP}}$- estimate (formally)

$$\ell_N(\theta) + \exp \log d\Pi_\gamma(\theta) \text{ "=" } -\frac{1}{2} \sum_{i=1}^N |Y_i - u_\theta(t_i, X_i)|^2 - e^{-2\|\theta\|^2_{H^\gamma_0}}$$

Penalised least squares or Tikhonov regularisation. Still non-convex. Instead, let us compute the posterior mean $E^\Pi[\theta | Z^{(N)}] = \tilde{\theta}_N$, then solve $u_{\tilde{\theta}}$ solving

$$\frac{\partial}{\partial t} u - \Delta u - f \circ u = 0 \quad \text{s.t. } u(0, \cdot) = \tilde{\theta}_N.$$

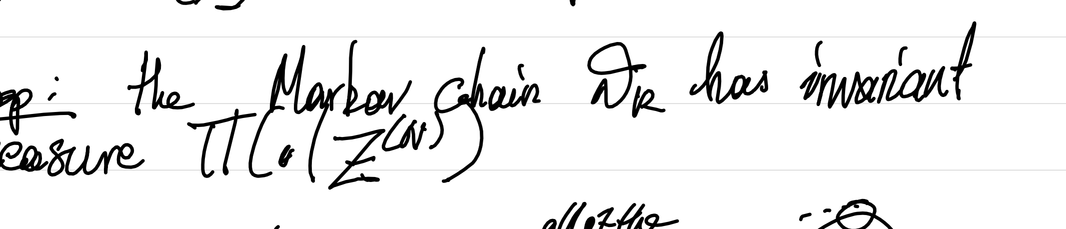
<u>Idea:</u> set up a Markov chain ϑ_k with invariant measure $\Pi(\cdot | Z^{(N)})$ and use
$$\frac{1}{K} \sum_{k=1}^{K} \vartheta_k \text{ to "estimate" } \tilde{\theta}_N.$$

<u>Example:</u> (pCN-algorithm). Start ϑ_0, step size $\delta > 0$. Compute

① $p_k = \sqrt{1 - 2\delta}\,\vartheta_{k-1} + \sqrt{2\delta}\,\xi$, $\xi \sim \Pi_\gamma$

② $\vartheta_k := \begin{cases} p_k & \text{with prob. } \min(1, e^{\ell_N(p_k) - \ell_N(\vartheta_{k-1})}) \\ \vartheta_{k-1} & \text{w.p. } 1 - \text{—"—} \end{cases}$
[on $\mathbb{R}^{D \le N}$]

<u>Prop:</u> the Markov chain ϑ_k has invariant measure $\Pi(\cdot | Z^{(N)})$



<u>Q's:</u> Suppose θ_0 is a ground truth initial condition, can we prove that
$$u_{\tilde{\theta}}, \quad u_{\frac{1}{K}\sum_{k=1}^K \vartheta_k} \quad \text{is close to } u_{\theta_0}$$
under $P^N_{\theta_0}$?

<u>Example</u> (ULA, MALA)

$\Theta \simeq \mathbb{R}^D$, $D \to \infty$.
Start at ϑ_0 and compute iterates
$$\vartheta_{k+1} = \vartheta_k - \delta \nabla \log \pi(\vartheta_k | Z^{(N)}) + \delta \xi_k,$$
$\xi_k \overset{iid}{\sim} N(0,1)$.

(approx) has invariant measure $\pi(\cdot | Z^{(N)})$, possibly after a MH adjustment.

II Posterior consistency for data assimilation

Aim for a posterior contraction result:

$$\pi(\theta : \|\theta - \theta_0\|_? > M\delta_N \mid Z^{(N)}) \xrightarrow[N\to\infty]{P_{\theta_0}^{\otimes N}} 0.$$

II.1 Helinger distance

Define $h^2(p_\theta, p_{\theta_0}) = \int \left(\sqrt{p_\theta} - \sqrt{p_{\theta_0}}\right)^2 dy\,dt\,dx.$

In our regression model we have

<u>Lemma</u> Suppose $\Theta_0 \subseteq \Theta$ s.t.

$$\sup_{\theta\in\Theta_0} \sup_{0\le t\le T} \|u_\theta(t,\cdot)\|_\infty \le U.$$

then $\exists\, C_U = \frac{1-e^{-U^2/2}}{2U^2}$ s.t.

$$C_U \|u_\theta - u_{\theta_0}\|^2_{L^2_T} \le h^2(p_\theta, p_{\theta_0}) \le \frac{1}{4}\|u_\theta - u_{\theta_0}\|^2_{L^2_T}.$$

$$\|H\|^2_{L^2_T} = \int_0^T \|H(t,\cdot)\|^2_{L^2(\Omega)}\,dt.$$

<u>Proof:</u> lecture notes Nickl '23.

One can show that $\exists$ a test $\Psi_N = \Psi(Z^{(N)}) = 1_{A_N}$ s.t.

$$E_{\theta_0}\Psi_N + \sup_{\substack{\theta\in\Theta_N\\ h(p_\theta,p_{\theta_0})>M\delta_N}} E_\theta(1-\Psi_N) \le e^{-cM^2N\delta_N^2}$$

"Type 1 error" universal

if $\log N(\Theta_N, h, \varepsilon) \le N\varepsilon^2 \quad \forall \varepsilon > 0$

↳ covering numbers.

<u>IDEA</u> of the proof: (post. contraction)

argue 1/denominator $\le e^{cN\delta_N^2}\,\Pi(\|u_\theta - u_{\theta_0}\|_{L^2_T} \le \delta_N)$

(one has then denom.)

<u>need:</u> $e^{cN\delta_N^2} E_{\theta_0} e^{\ell_N(\theta) - \ell_N(\theta_0)}(1-\Psi_N + \Psi_N)$

+ change of mean...

II.2 The prior on $C([0,T]; L^2(\Omega))$.

<u>Lemma:</u> Let $\Theta_0 \in H^\gamma$, let $\Pi = \Pi_\gamma$ be $\mathcal{N}(0, \rho^2(-\Delta)^{-\gamma})$ where $\rho = \rho_N = \frac{1}{\sqrt{N}\,\delta_N}$, $\delta_N = N^{-\frac{\gamma}{2\gamma+d}}$, $\rho_N \to 0$.

Then, there exist $A, c > 0$ s.t. for reaction diffusion with $f \in C_c^\infty(\mathbb{R})$, or Navier Stokes we have:

$$\Pi(\theta : \sup_{0\le t\le T}\|u_\theta\|_\infty \le U, \|u_\theta - u_{\theta_0}\|_{L^2_T} < \delta_N) \ge e^{-AN\delta_N^2}$$

and for $0 < \beta < \gamma - d/2$, and M large enough

$$\Pi(\theta : \|\theta\|_{H^\beta} \le M, \theta = \theta_1 + \theta_2, \|\theta_1\|_{H^\gamma} \le M_1, \|\theta_2\|_{L^2} \le M\delta_N) \ge 1 - e^{-c_M N\delta_N}$$

<u>Proof:</u> 2) $\theta = \frac{1}{\sqrt{N}\,\delta_N}\theta'$, $\theta' \in H^\beta$ a.s.

(use Fernique thm + Borell's iso-ineq. with RKHS $= \sqrt{N}\,\delta_N H^\gamma$).

<u>For the first:</u> since $u_{\theta_0} \in L^\infty([0,T]; H^\gamma)$, it suffices to prove ① with $\|u_\theta - u_{\theta_0}\|_\infty$. We have for $\|\theta\|_{H^\xi} + \|\theta_0\|_{H^\xi} \le U$ for some $\xi > d/2$ the regularity estimate

$$\|u_\theta - u_{\theta_0}\|_{L^2_T,\, L^\infty_T} \le C_{U,T}\|\theta - \theta_0\|_{L^2,\, H^\xi}$$

The prob. in question is

$$\ge \Pi(\theta : \|\theta - \theta_0\|_{H^\xi} \le \frac{U}{C_U}, \|\theta - \theta_0\|_{L^2} \le \frac{\delta_N}{C_U})$$

C-M

$$\ge e^{-N\delta_N^2\|\theta_0\|^2_{H^\gamma}}\,\Pi(\|\theta'\|_{H^\xi} \le \sqrt{N}\,\delta_N\frac{U}{C_U}, \|\theta'\| \le \frac{\sqrt{N}\delta_N^2}{C_N})$$

Gaussian

$$\ge e^{-cN\delta_N^2}\,\underbrace{\Pi(\|\theta'\|_{H^\xi} \le \underbrace{\sqrt{N}\,\delta_N}_{\to\infty}\frac{U}{C_U})}_{\to 1}\cdot\Pi(\|\theta'\|_{L^2} \le \frac{\sqrt{N}\delta_N^2}{C_N})$$

Linde + Li, AoP, 1999

$$\ge e^{-AN\delta_N^2}$$

□

From what precedes we have shown

$$\pi(\theta : \|\theta\|_{H^\beta} \le M, \|u_\theta - u_{\theta_0}\|_{L^2_T} \le M\delta_N \mid Z^{(N)}) \xrightarrow[N\to\infty]{Pr} 1$$

Also, by UI, $\|u_{\tilde\theta} - u_{\theta_0}\|_{L^2_T} = O_p(\delta_N)$, $\tilde\theta = E^\Pi[\theta \mid Z^{(N)}]$

II.3 Stability estimates

For 2d NS, we can obtain a differential inequality for $\Phi(t) = \frac{\|\nabla w(t)\|^2_{L^2}}{\|w(t)\|^2_{L^2}}$ for $w = u_\theta - u_{\theta_0}$, an estimate

<u>Theorem:</u> For $\|\theta\|_{H^\xi} + \|\theta_0\|_{H^\xi} \le U$, and $u_\theta(t), u_{\theta_0}(t)$ solutions to 2d NS. Here $\exists\, C_{U,T}$ s.t.

$$\|\theta - \theta_0\|_{L^2} \lesssim C_{U,T}\left(\log\frac{C_U}{\|u_\theta(t) - u_{\theta_0}(t)\|_{L^2}}\right)^{-1/2} \quad \forall t > 0$$

& so for $\|u_\theta(t) - u_{\theta_0}(t)\|_{L^2}$ replaced by $\|u_\theta - u_{\theta_0}\|_{L^2_T}$.

<u>Corollary:</u> (Consistency of data Assimilation)

$$\Pi(\theta : \sup_{0\le t\le T_0}\|u_\theta(t) - u_{\theta_0}(t)\|_{L^2} > \frac{\Pi}{\sqrt{\log N}} \mid Z^{(N)}) \xrightarrow{P_{\theta_0}^{\otimes N}} 0.$$

<u>Remark:</u> $\exists\, \theta_j$ s.t. log-modulus is <u>sharp</u>, for $t > 0$.

<u>Theorem:</u> (Reaction diffusion). Assume $\|\theta\|_{H^1} + \|\theta_0\|_{H^1} \le U$. then

$$\int_0^T \|u_\theta(t) - u_{\theta_0}(t)\|^2_{L^2(\Omega)}\,dt \gtrsim C_{U,T}\|\theta - \theta_0\|^2_{H^{-1}}$$

<u>Proof:</u> Take $w = u_\theta - u_{\theta_0}$ which solves

$$\left(\frac{\partial}{\partial t} - \Delta u\right)w = f(u_\theta) - f(u_{\theta_0})$$

$$= f'(\tilde u)w$$

$$= \tilde V(t)w$$

with initial cond. $\theta - \theta_0$

Then w compares to the solution $\bar w_\varepsilon$ to the PDE $\left(\frac{\partial}{\partial t} - \Delta\right)\bar w_\varepsilon - V_\varepsilon w_\varepsilon$ where $V_\varepsilon = f'(\tilde u(0))$ on $[0,\varepsilon]$.

Then on $[0,\varepsilon]$, V_ε is time-independent so the LHS of (†) is

$$\gtrsim \int_0^\varepsilon \|w_\varepsilon\|^2_{L^2}\,dt \text{ and}$$

$$w_\varepsilon = \sum_{j=1}^\infty e^{-t\lambda_j}\langle e_j, \theta - \theta_0\rangle e_j,$$

where (e_j, λ_j) are s.t. $(\Delta - V_\varepsilon)e_j = -\lambda_j e_j$.

so (†)

$$= \int_0^\varepsilon \sum_{j=1}^\infty e^{-2t\lambda_j}\langle e_j, \theta - \theta_0\rangle^2_{L^2}\,dt$$

$$= \sum_{j=1}^\infty \frac{1}{2\lambda_j}(1 - e^{-\varepsilon\lambda_j})\langle e_j, \theta - \theta_0\rangle^2_{L^2}$$

$$\gtrsim \|\theta - \theta_0\|^2_{H^{-1}}.$$

□

Now, $L^2 \subseteq [H^\beta, H^{-1}]$ and so by interpolation we deduce

$$\Pi(\theta : \|\theta\|_{H^\beta} \le M, \|\theta - \theta_0\|_{L^2} \le \underbrace{M\delta_N^{\frac{\beta}{\beta+1}}}_{\tilde\delta_N} \mid Z^{(N)}) \xrightarrow[N\to\infty]{P_{\theta_0}^{\otimes N}} 1$$

where $\tilde\delta_N = N^{-\frac{\gamma}{2\gamma+d}\frac{\beta}{\beta+1}}$ $\left(\simeq \frac{1}{\sqrt{N}}, \gamma, \beta \to \infty\right)$.

III Bernstein–von Mises theorems

If $\Theta = \mathbb{R}^p$, p is fixed, $d\Pi > 0$ on $\mathbb{R}^p$, $I_N(\theta_0)$ ... Fisher inf. of model, then

$$\left\|\Pi(\cdot \mid Z^{(N)}) - N_{\mathbb{R}^p}(\tilde\theta, \tfrac{1}{N}I(\theta_0)^{-1})\right\|_{TV} \xrightarrow{P_{\theta_0}^{\otimes N}} 0$$

Freedman (1999) $\Rightarrow$ BvM <u>fails</u> in ∞-dim in ℓ_2.

Castillo / Nickl (2013/14) ... ✓ ... in H^{-k}, $k > d/2$

Have shown

$$\sup_{0\le t\le T_p} \|u_{\tilde\theta}(t,\cdot) - u_{\theta_0}(t,\cdot)\|_{L^2} = O_{P^N_{\theta_0}}(\tilde\delta_N) \quad \lesssim N^{-c},\ c < 1/2.$$

III.1 Main result

The laws $\hat\Pi_N$ induce Borel probability measures on

$$\mathcal{C} = C([t_{min}, t_{max}]; C(\Omega)), \ \|\cdot\|_\infty,$$
$$0 < t_{min} < t_{max}.$$

To metrise distance between prob. measures μ, ν on $\mathcal{C}$ we take

$$W_{1,\mathcal{C}}(\mu,\nu) = \sup_{\substack{H:\mathcal{C}\to\mathbb{R}\\ \|H\|_{Lip}\le 1}} \left|\int_{\mathcal{C}} H(x)(d\mu(x) - d\nu(x))\right|$$

<u>Theorem</u> (N24, KNR25) Let $\theta_0 \in H^\gamma_0 = H^\gamma \cap L^2_0$, $\gamma > ?$
Then $W_{1,\mathcal{C}}\left(\mathrm{Law}(\sqrt{N}(u_\theta - u_{\tilde\theta_N}) | Z^{(N)}), \mathrm{Law}(\mathcal{U})\right)$

$$\xrightarrow[N\to\infty]{P^N_{\theta_0}} 0.$$

and $\sqrt{N}(u_{\tilde\theta_N} - u_{\theta_0}) \xrightarrow[\mathcal{C}]{d} \mathrm{Law}(\mathcal{U})$,

where $\mathcal{U}$ is the Gaussian random field in $\mathcal{C}$, solving the linear PDEs

$$(\ddagger) \quad \frac{\partial}{\partial t}\mathcal{U} - \Delta\mathcal{U} = f'(u_{\theta_0})\,\mathcal{U} \quad (\text{r. diffusion})$$

$$\text{or } = \left(\frac{\partial}{\partial t} - P\Delta\right)\mathcal{U} + B(u_{\theta_0}, \mathcal{U}) + B(\mathcal{U}, u_{\theta_0})$$

(natural lin. of flows of PDE)

with initial condition (analogous to inv. Fisher inf. in fin. dim.)

$$\mathcal{U}(0,\cdot) \sim X \sim N_{\theta_0} = N(0, (\mathbb{I}^*_{\theta_0}\mathbb{I}_{\theta_0})^{-1})$$

defining a Borel prob. meas. on H^{-k} for $k > 1+d/2$ (universal, γ suff. large).

II. Information operators.

For $h \in L^2_0$ let $\mathbb{I}_{\theta_0}(h) = \mathcal{U}_h$ be the sol^n to the PDE $(\ddagger)$ with initial condition h. Then one shows

$$\|u_{\theta_0+h} - u_{\theta_0} - \mathbb{I}_{\theta_0}(h)\|_{L^2} = O(\|h\|^2_{L^2}),$$ so $\mathbb{I}_{\theta_0}$ is the 'score' operator linearising $\theta \mapsto u_\theta$. If $\mathbb{I}_{\theta_0}: L^2_0 \to L^2_T$ has adjoint $\mathbb{I}^*_{\theta_0}: L^2_T \to L^2_0$, then the Fisher information operator is

$$\boxed{\mathbb{I}^*_{\theta_0}\mathbb{I}_{\theta_0} : L^2_0 \to L^2_0.}$$

<u>Theorem</u>: For $\eta \ge 0$, the operator $\Delta\mathbb{I}^*_{\theta_0}\mathbb{I}_{\theta_0}$ is a homeomorphism of $H^\eta_0 := H^\eta \cap L^2_0$.

<u>Proof</u>: <u>Idea</u>: show first that $\mathbb{I}^*\mathbb{I}$ and I^*I where $\mathcal{U}_h = I(h)$ solves

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta\right)\mathcal{U} = 0 \\ u(0,\cdot) = h \end{cases} \quad \text{are s.t.}$$

$\Delta(\mathbb{I}^*\mathbb{I} - I^*I)$ is a compact operator on H^η_0.

To apply a Fredholm argument, let us compute

$$\Delta I^* I = \Delta\int_0^T I^*_t(I_t(h))\,dt$$
$$= \Delta\sum_{j=1}^\infty \int_0^T e^{-2t\lambda_j}\langle e_j, h\rangle e_j$$
$$= \sum_{j=1}^\infty (-\lambda_j)\frac{1}{(-2\lambda_j)}\left[e^{-2T\lambda_j} - 1\right]\langle e_j, h\rangle e_j$$
$$= -\frac{1}{2}\mathrm{Id} + \underbrace{\mathcal{U}_h(2T)}_{=K}$$

So overall $\Delta\mathbb{I}^*\mathbb{I} = -\frac{1}{2}\mathrm{Id} + K$ so let us show it is also injective. First assume $\Delta\mathbb{I}^*\mathbb{I}h = 0 \Rightarrow \mathbb{I}^*\mathbb{I}h = 0$. So

$$0 = \langle \mathbb{I}^*\mathbb{I}h, h\rangle_{L^2} = \langle \mathbb{I}h, \mathbb{I}h\rangle_{L^2} = \|\mathbb{I}h\|^2_{L^2} \gtrsim \|h\|_{H^{-1}}$$

$\Rightarrow h = 0$ (as in the non-linear stability estimates).

$\square$

III. BvM for initial conditions

Following N29, we prove

<u>Theorem</u>: We have for $k > 2d+3$

$$W_{1,H^{-k}}\left(\mathrm{Law}(\sqrt{N}(\theta - \tilde\theta_N) | Z^{(N)}), N_{\theta_0}\right) \xrightarrow[N\to\infty]{P_{\theta_0}} 0$$

and $\sqrt{N}(\tilde\theta_N - \theta_0) \xrightarrow{d} N_{\theta_0}$.

<u>Rmk</u>: $k > 1+d/2$ is necessary!

<u>Proof</u>: (idea)

1. localise to $\Pi^{D_N}(\cdot | Z^{(N)})$ where $\Pi^{D_N} = \frac{\Pi(\cdot \cap D_N)}{\Pi(D_N)}$

   where $D_N = \{\|\theta\|_{H^\beta} \le M, \|\theta - \theta_0\|_{L^2} \le M\delta_N\}$

2. Given $\psi \in H^k_0$, take $\Psi = (\mathbb{I}^*_{\theta_0}\mathbb{I}_{\theta_0})^{-1}\psi \in H^{k-2}_0$

3. Study Laplace transform

$$E^{\Pi^{D_N}}\left[e^{t\sqrt{N}(\langle\theta,\psi\rangle - \hat\Psi_N)} | Z^{(N)}\right]$$
$$= \frac{\int_{D_N} e^{t\sqrt{N}(\langle\theta,\psi\rangle - \hat\Psi_N) + \ell_N(\theta) - \ell_N(\theta_t) + \ell_N(\theta_t)}\,d\pi(\theta)}{\int_{D_N} e^{\ell_N(\theta)}\,d\pi(\theta)} \quad \rightarrow \text{Ch3 in GNK}$$

where $\theta_t = \theta - \frac{t}{\sqrt{N}}\Psi$

$$\hat\Psi_N = \langle\theta_0, \psi\rangle - \frac{1}{N}\sum_{i=1}^N \varepsilon_i\,\mathbb{I}_{\theta_0}(\Psi)(t_i, X_i)$$
$$= e^{\frac{t^2}{2}\|\mathbb{I}_{\theta_0}\Psi\|^2_{L^2} + o_P(1)} \cdot \underbrace{\frac{\int e^{\ell_N(\theta_t)}\,d\pi(\theta)}{\int e^{\ell_N(\theta)}\,d\pi(\theta)}}_{\to 1}$$

4. $\frac{d\Pi(\theta_t)}{d\pi(\theta)} \to 1$.

5. To prove convergence in function space by fin. dim. conv. + tightness in $H^{-k}_0 = (H^k_0)^*$.

6. U.I. + conv. of moments.

III.4. to prove the main theorem, we use

$$\sqrt{N}(u_{\tilde\theta_N} - u_{\theta_0}) = \underbrace{Du_{\theta_0}}_{\text{linearisation}}(\sqrt{N}(\tilde\theta_N - \theta_0)) + \sqrt{N}\|\tilde\theta_N - \theta_0\|^2_{L^2}$$
$$= \mathbb{I}_{\theta_0}(\sqrt{N}(\tilde\theta - \theta_0)) + O_p(1), \text{ use } \mathbb{I}_{\theta_0}: H^{-k} \to \mathcal{C}$$
$$\xrightarrow{d} \mathbb{I}_{\theta_0}(X), \quad X \sim N_{\theta_0}.$$

III.5 Cramer-Rao lower bounds

Recall Gauss-Markov theorem:
Estimator $E(\tilde\theta - \theta_0)^2 = \mathrm{Var} + \cancel{\mathrm{Bias}^2}$ for consistent estimators asymptotically.

<u>One shows</u>

<u>Thm</u>: For $\theta_0 \in H^\gamma$ then

$$\liminf_{N\to\infty}\ \inf_{\hat\theta(Z^{(N)})}\ \sup_{\substack{h\in H^1\\ \|h\|_{H^1}\le 1}} N E_{\theta_0 + \frac{h}{\sqrt{N}}}\|u_{\hat\theta} - u_{\theta_0}\|^2_{\mathcal{C}}$$
$$\ge E\|\mathcal{U}\|^2_{\mathcal{C}}$$

and this lower bound is attained by $u_{\tilde\theta}$ (posterior mean).

Examples Session:

Set $\Omega = \mathbb{T}^2$

$H^\beta_0 = H^\beta \cap \{ f : \int_\Omega f \, dx = 0, \nabla \cdot f = 0 \}$

$$\frac{\partial u}{\partial t} - \nu \Delta u + B(u,u) = f, \quad u(0) = \theta.$$

$$B(u,u) = P((u \cdot \nabla)(u)), \quad P : L^2 \to L^2_0.$$

$$(\text{prior}) \ \Pi_\gamma = \mathcal{N}(0, \rho_\gamma^2 (-\Delta)^{-\gamma}) \rightsquigarrow \underset{(\text{posterior})}{\Pi(\cdot \mid Z^{(N)})}$$

<u>Have contraction at trajectory level:</u>

$$\Pi\big(\theta \in H^1_0 : \|\theta\|_\beta \le M, \|u_\theta - u_{\theta_0}\|_{L^2_T} \le M \overset{\to 0}{\delta_N} \mid Z^{(N)}\big) \xrightarrow[N \to \infty]{P_\infty^{\otimes N}\text{-prob.}} 1$$

Want <u>Transfer of:</u>

$$\|u_\theta - u_{\theta_0}\|_{L^2_T} \text{ small} \overset{?}{\Rightarrow} \|\theta - \theta_0\|^\alpha_{L^2} \le \|u_\theta - u_{\theta_0}\|_{L^2_T} \le M\delta_N$$

For $w = u_\theta - u_{\theta_0}$ solves:

$$\begin{cases} \frac{\partial w}{\partial t} - \nu \Delta w + B(u_\theta, u_\theta) - B(u_\theta, u_{\theta_0}) = 0 \\ w(0) = \theta - \theta_0. \end{cases}$$

<u>Rearrange to get: (linearised PDE)</u> $\frac{\partial w}{\partial t} - \nu \Delta w + B(u_\theta, w) + B(w, u_{\theta_0}) = 0$

Take L^2-inner product with $(-\Delta^{-1})w$ (take H^{-1} inner product).

<u>Get:</u> $$\left\langle \frac{\partial w}{\partial t}, (-\Delta)^{-1} w \right\rangle_{L^2} - \nu \left\langle \Delta w, (-\Delta)^{-1} w \right\rangle_{L^2} + \left\langle B(u_\theta, w) + B(w, u_{\theta_0}), (-\Delta)^{-1} w \right\rangle_{L^2} = 0$$

(any power of $(-\Delta)$ is self-adj. on $L^2(\mathbb{T})$).

$$\le C(\|u_\theta\|_{H^2} + \|u_{\theta_0}\|_{H^2}) \times \|w\|_{H^{-1}} \cdot \|w\|_{L^2}$$

$<\infty$ uniformly in θ on some ball.

$\hookrightarrow$ Navier-Stokes dependent.

$$\Rightarrow \left\langle \frac{\partial}{\partial t} (-\Delta)^{-1/2} w, (-\Delta)^{-1/2} w \right\rangle_{L^2}$$

$$0 \le \frac{1}{2} \frac{d}{dt} \|(-\Delta)^{1/2} w\|^2_{L^2} + \nu \|w\|^2_{L^2} + c\|w\|^2_{L^2}.$$

Fix $s > 0$, integrate in $t \in [0, s]$

$$\Rightarrow -\frac{1}{2} \|w(0)\|^2_{H^{-1}} \le \frac{1}{2} \|w(s)\|^2_{H^{-1}} + (\nu + c) \int_0^s \|w(t)\|^2_{L^2}$$

Poincaré $\le \|w(s)\|^2_{L^2}$ $(\delta \cdots \lesssim \|w\|_{L^2})$

integrate in $s \in [0, T]$:

$$\Rightarrow -\frac{1}{2} T \underbrace{\|w(0)\|^2_{H^{-1}}}_{\|\theta - \theta_0\|^2_{H^{-1}}} \le \frac{1}{2} \int_0^T \|w(s)\|^2_{L^2} ds + \int_0^T \underbrace{(T-s)}_{\le T} \|w(s)\|^2_{L^2} ds$$

$\square$.

Recall $W_{1, H^{-k}}(\text{Law}(\sqrt{n}(\theta - \tilde{\theta}_n)), \mathcal{N}_{\theta_0}) \xrightarrow[N \to \infty]{P_\infty^{\otimes N}} 0$

$\mathcal{C} := C([t_{min}, t_{max}]; C(\Omega))$

$\theta \sim \Pi(\cdot / Z^{(N)})$

$\sigma^2_N = E^\Pi[\theta \mid Z^{(N)}]$

Define a centred Gaussian process $\{W(f) : f \in \mathcal{C}^\infty_0\}$ with covariance $\mathbb{E}[W(f) W(g)] = \langle f, (I^* I)^{-1} g \rangle_{L^2}$.

Restrict to the eigenfunctions of $-P\Delta = -\Delta$, $\{e_j\}_{j \ge 1} \to \{W(e_j) : j \ge 1\}$ defines a cylindrical prob. meas. on $\mathbb{R}^{\mathbb{N}}$, $\mathcal{N}_{\theta_0}$ as the law of $(W(e_1), W(e_2), \ldots)$

Fix $\beta \in \mathbb{R}$

$$\mathbb{E}[\|Z\|^2_{H^{-\beta}}] = \mathbb{E}\Big[\sum_{j \ge 1} \lambda_j^{-\beta} |W(e_j)|^2\Big]$$

$$\overset{\checkmark}{=} \sum_{j \ge 1} \lambda_j^{-\beta} \underbrace{\langle e_j, (I^* I)^{-1} e_j \rangle_{L^2}}$$

$$\lesssim \sum_{j \ge 1} \lambda_j^{1-\beta} \qquad \le \|e_j\|_{H^2} \cdot \underbrace{\|(I^* I)^{-1} e_j\|_{H^{-1}}}$$

$(I^* I : H^\ell_0 \to H^{\ell+2}_0 \ \forall \ell \ge 2)$ $\quad \lesssim \|e_j\|_{H^1}$

$\lesssim \|e_j\|^2_{H^2}$

$= \|(-\Delta)^{1/2} e_j\|^2_{L^2}$

$= \lambda_j$.

<u>Weyl's asymptotics:</u> $\lambda_j \simeq j^{2/d} \overset{d=2}{=} j$ (on bdd. domains).

So $\mathbb{E}[\|Z\|^2_{H^{-\beta}}] \lesssim \sum_{j \ge 1} j^{-(\beta - 1)} < \infty$

$$\Leftrightarrow \beta - 1 > 1 \Leftrightarrow \beta > 2.$$

This is sharp due to <u>exactness</u> of Weyl asymptotics.