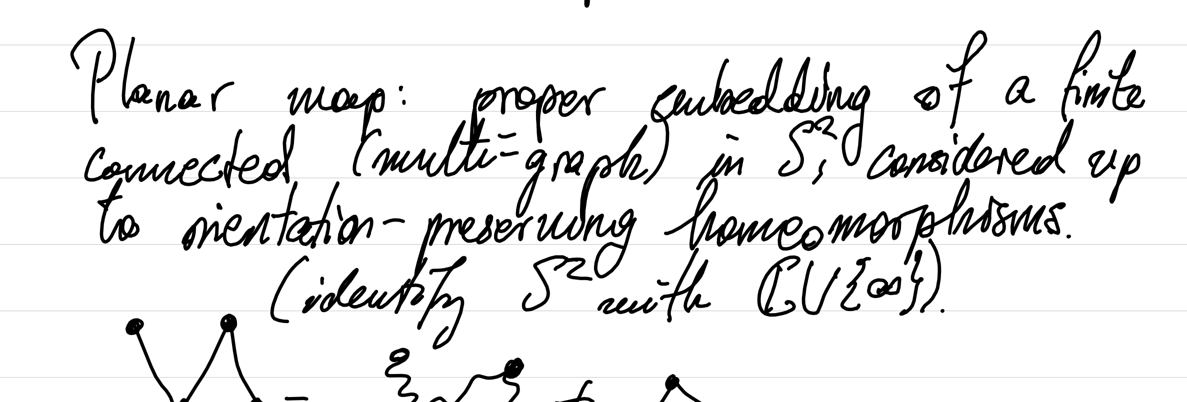
Scaling Limits of Random Planar Maps

Planar map: proper embedding of a finite connected (multi-graph) in S^2, considered up to orientation-preserving homeomorphisms. (identify S^2 with $\mathbb{C}\cup\{\infty\}$).



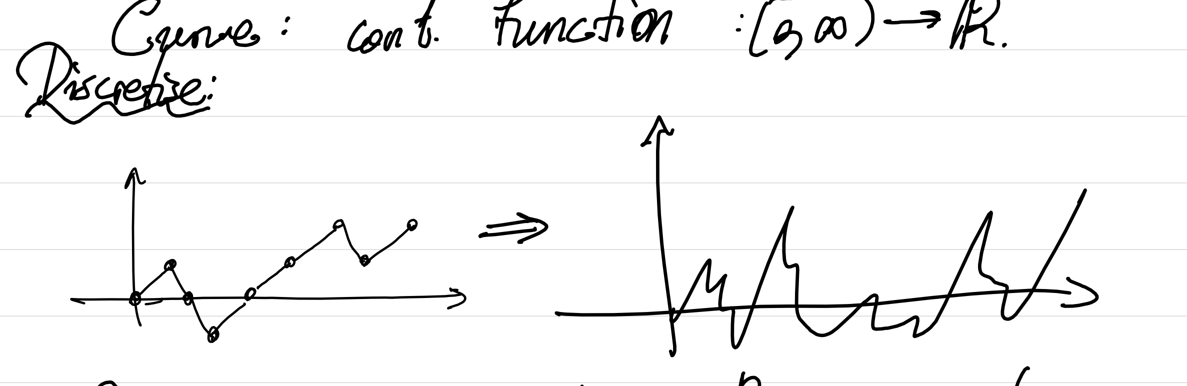
Oriented root edge e.
Vertices $V(M)$, edges $E(M)$, faces $F(M)$.
Root face to the right of e

Combinatorics
Geometry
Physics (field thy, string thy) } natural model
Probability } for "random surface."

Random Planar Maps (RPMs)

Example 1: (uniform RPM).
$S_n = \{m : m \text{ is pl. map w/n edges}\}$
$\forall m \in S_n$
$\mathbb{P}(M=m) = (\# S_n)^{-1}$

Motivation (probabilistic)
Q: What is a uniform curve?
Curve: cont. function $:[0,\infty)\to\mathbb{R}$.
Discretise:



Is there an analogue for surfaces?
Q: What is a uniform surface?
A: after discretising space, take uniform RPM.

Example 2: (tree-weighted RPM).

Spanning tree: for m is a set of edges which is spanning, connected and has no cycles.

M, $T(M)=5$

$T(m) = \#$ spanning trees admitted by m.
$\forall m \in S_n$
$\mathbb{P}(M=m) \propto T(m)$.

Equivalently, pick uniform pair (M_n, T), then forget tree
RPM C_n, spanning tree

Motivation for example 2:
- $T(m) = \det \Delta(m)$ ["Laplacian determinant"] ↳ discrete Laplacian obtained from adjacency mat.
& related to other statistical physics models on RPM
- Scaling limits
- Mullin bij. (1967) $(M,T) \xleftrightarrow{\text{(bijection)}}$ walk $(W_j)_{j\in[2n]}$ on $\mathbb{Z}_+^2$
$\# E(M) = n$, $W_0 = W_{2n} = 0$, $\|W_{j+1} - W_j\|_1 = 1$

What does it mean that a RPM is conv. in the scaling limit?

Convergence of random variables:

Metric space (S,d)
Rand var: $X_1, X_2, \dots$, w/values in (S,d)
$(X_n)_n$ conv. in law to X if for all cont. $f: S\to[0,1]$, $E[f(X_n)] \Rightarrow E[f(X)]$,
$X_n \Rightarrow X$.

Q: What is the most natural choice for (S,d)? Different choices for (S,d) gives different topologies of convergence.

3 top. in context of RPM:

- metric topology (graph distance)
- peanosphere topology (statistic of stat. phys. model converges)
- Conformal topology

Metric top:

$S = \{X : X \text{ compact metric space}\}$
$d = d_{GH}$, $S\times S \to [0,\infty)$
GH: Gromov-Hausdorff
$d_{GH}(X,Y)=0$ if X and Y are isometric.
↳ quantifies how much one needs to "distort" X to get Y.

Thm (LeGall '13, Miermont '13, Bettinelli-Jacob-Miermont '14)
Uniform random planar maps $\Rightarrow$ Brownian map ← random metric space
in metric top.

Rmk: Conv. of non-uniform RPMs open.

Rmk: peanosphere top. conv. is known for uniform & non-uniform RPMs

Planar maps w/ disk topology:

Boundary of map = boundary of root face

∂ non-simple. ∂ simple.

Asymptotically, ∂ does not matter → same scaling limit.

Conformal embedding of RPMs:

$M_1, M_2, M_3, \dots$ RPMs in discrete top., $\# E(M_n) = n$

Discrete conformal topology:
$\phi_n : V(M_n) \to \mathbb{D}$

Rescaled vertex counting measure: $(A\subset V(M_n))$
$\mu_n(A) = \frac{1}{n}\# A$
atoms of μ_n'
$\mu_n' = \phi_{n*}\mu_n$, atomic vertex on $\mathbb{D}$.

Let $S = \{\mu : \mu \text{ is a finite Borel measure on } \mathbb{D}\}$

Levy-Prokhorov metric: (Separable)
$d = d_{LP}: S\times S \to [0,\infty)$
$d_{LP}(\mu,\nu) = \inf\{\varepsilon>0 : \mu(A) \le \nu(A^\varepsilon)+\varepsilon,\ \nu(A)\le\mu(A^\varepsilon)+\varepsilon\}$
$A^\varepsilon = \varepsilon$-nbhd of A.

(M_n) conv. in conformal top. (or under discrete conformal embedding) if $\exists$ random μ s.t. $\mu_n \Rightarrow \mu$.

Conj: A number of RPM converge to γ-Liouville quantum gravity (LQG) under discrete conformal embedding. (Lots of ways of doing?)

Thm A (Gwynne-Miller-Sheffield '21): Conj. is true for mated-CRT map, Tutte embedding, $\gamma\in(0,2)$.

Thm B (H-Sun '23): Conj. true for uniform triangulations, Cardy-Smirnov embedding, $\gamma=\sqrt{8/3}$.

Thm C (H '25): Conj. true for tree-weighted maps, Tutte embedding, $\gamma=\sqrt{2}$.

Q: What is a discrete conformal embedding?
What is a conformal map?

$D \xrightarrow{\phi} \tilde{D}$

Suppose $\phi: D \to \tilde{D}$ orientation-preserving, smooth, bijective; $D, \tilde{D}$ Jordan domains.

Equivalent conditions:
ⓐ complex differentiable (w/ non-zero derivative)
ⓑ Cauchy-Riemann eqns.
ⓒ $\phi\circ W$ is 2D BM, if W is 2D BM (up to time change)

$D \xrightarrow{\phi} \tilde{D}$, W, $\phi\circ W$

ⓓ $\phi\circ\Gamma$ is percolation scaling limit if Γ is percolation scaling limit.

Tutte embedding: discretises def ⓒ of a conformal map

$D \xrightarrow{\phi} \mathbb{D}$, a, b, 0, 1

$\exists!$ conformal $\phi: D\to\mathbb{D}$ s.t. $\phi(a)=0$, $\phi(b)=1$.
Discretise domain:

$1/n$, b_n, ∂D_n, a_n, D_n, $\mathbb{D}$

Goal: define "discrete conformal" $\phi_n : D_n \cup \partial D_n \to \mathbb{D}$ s.t. $\phi_n \approx \phi$
Def $\phi_n(a_n) = 0$, $\phi_n(b_n) = 1$.

vertex

λ^v_n = law of $\tilde\phi_n \circ W^v|_{[0,\tau_n]}$

$\lambda^v_{\omega_n}$ = law of $\tilde\phi_n \circ W^v|_{[0,\tau_n]}$ given ω_n.

λ^x = law of $Z^x|_{[0,\tau]}$

<u>Annealed convergence:</u>

$$d_{\text{Prokhorov}}(\lambda^{an}_n, \lambda^0) \xrightarrow{n\to\infty} 0$$

<u>Quenched conv:</u> (stronger)

$$d_{\text{Prokhorov}}(\lambda^{an}_{\omega_n}, \lambda^0) \underset{n\to\infty}{\Rightarrow} 0$$

<u>Prop:</u> suppose

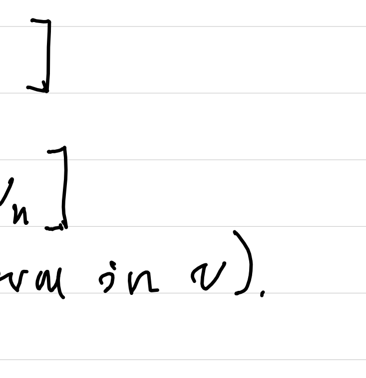
(I) $\sup_{v\in V(M_n)} d_{\text{Prok}}(\lambda^v_{\omega_n}, \lambda^{\tilde\phi_n(v)}) \Rightarrow 0$.

(II) $\tilde\phi_n(v) \in \partial D \quad \forall v \in \partial V(M_n)$

(III) If $v_1, v_2, \dots$ are ordered counter clockwise on ∂M_n then $\tilde\phi_n(v_1), \tilde\phi_n(v_2), \dots$ are ordered counter-clockwise on ∂D.

Then $\|\tilde\phi_n - \phi_n\|_\infty$.

<u>Rmk:</u> Conditions (II), (III) are necessary (can produce counter-examples with rescaling, or consider $\tilde\phi_n(v) = (\phi_n(v))^2$ (III) still conformal shows that (III) is necessary).



<u>Proof:</u> (Sketch) Consider

case (i) $v \in \partial V(M_n)$

$\mathbb{P}[Z^0(\tau) \in \text{arc}_{\partial D}(1, \tilde\phi_n(v)) \mid \omega_n]$

$\overset{\text{(I)}}{\approx} \mathbb{P}[\tilde\phi_n(W^{a_n}(\tau_n)) \in \text{arc}_{\partial D}(1, \tilde\phi_n(v)) \mid \omega_n]$

$\overset{\text{(II) \& (III)}}{=} \mathbb{P}[W^{a_n}(\tau_n) \in \text{arc}_{M_n}(b_n, v) \mid \omega_n]$

$\overset{\text{Tutte defn}}{=} \mathbb{P}[Z^0_\tau \in \text{arc}_D(1, \phi_n(v)) \mid \omega_n]$

$\Rightarrow \tilde\phi_n(v) \approx \phi_n(v)$ (error uniform in v).

<u>Case (ii):</u> $v \in V(M_n) \setminus \partial V(M_n)$

$$\mathbb{E}[Z^{\tilde\phi_n(v)}(\tau)] \overset{\text{OST}}{=} \tilde\phi_n(v)$$

$\approx$ (I)

$\mathbb{E}[\tilde\phi_n(W^v(\tau_n))]$

$\approx$ case (i)

$\mathbb{E}[\phi_n(W^v(\tau_n))]$

$\|$ Tutte def.

$\phi_n(v)$

$\square$

<u>Rmk:</u> similar prop. holds for percolation & Cardy-Smirnov embedding.

Random graph (V,E), $V \subseteq \mathbb{R}^2$ locally finite, s.t.

- translation invariant and modulo scaling "allowed to scale by rand. const. before & after scaling".

- ergodic (modulo scaling)
- well-connected
- high-degree vert & long edges unlikely.

Then (GMS):

SRW on $(V, M) \Longrightarrow$ 2D BM quenched.

<u>Our (V,E):</u>

Consider $\tilde\phi_n$-emb. of RPM. Pick $z_0 \sim$ Leb and Zoom in near z_0 while $n \to \infty$.

<u>Thm:</u> local limit exists and satisfies assumptions of Thm GMS.

Recall defn of $\tilde\Phi_n$

$$\begin{array}{ccc} t_v \ \text{Walk} & \xrightarrow{n\to\infty} & \text{2D BM} \ t \\ \updownarrow \text{Mullin} & & \downarrow \text{Mating of trees} \\ N \ (M, T) & & \sqrt{2}\text{-LQG} \quad \rho(t) \in \overline{D} \end{array}$$

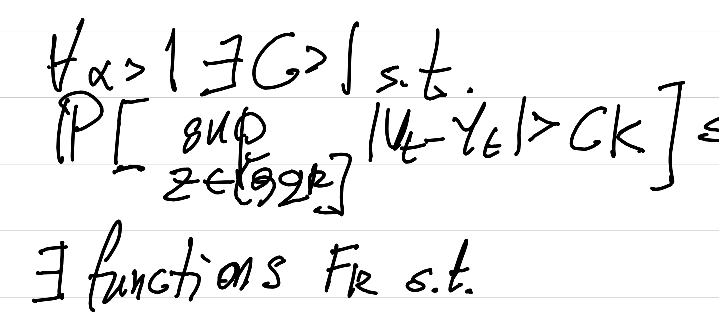
$$\tilde\phi_n(v) = \rho(t_v).$$

<u>Strong couplings of 2D random walk & 2D BM</u>

(KMT-type, Komlós-Major-Tusnády '75).

For $a > 0$, coupling of

- $(V_t)_{t\geq 0}$, $V_t = P_t - at$, P = rate a Poisson process.



- $(Y_t)_{t\geq 0}$ $\text{Var}\ a$ 1-D BM.

(a) $\forall \alpha > 1\ \exists C > 1$ s.t.
$$\mathbb{P}\Big[\sup_{t\in[0,2^k]} |V_t - Y_t| > Ck\Big] \leq 2^{-\alpha k}$$

(b) $\exists$ functions F_k s.t.

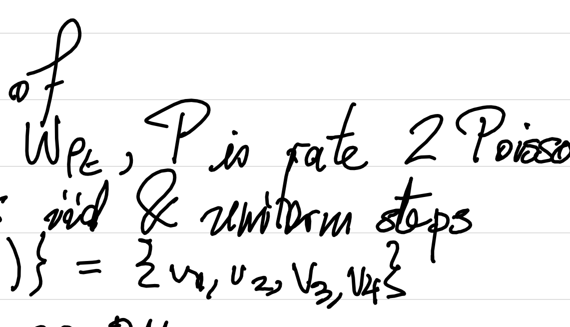
$$\mathbb{P}[V|_{[0,2^k]} \neq F_k(Y|_{[0,2^{k+m}]})] \leq 2^{-\varepsilon m}$$

"Y is determining V in a local sense w. very high prob."

<u>Local limit of coupling of Pollard's coupling for processes on $[0,1]$.</u>

Define $A = \{t \geq 0 : \Delta V_t \neq 0\}$

Given $\#(A \cap [0, 2^k])$, determine $\#(A \cap [0, 2^{k-1}])$ & $\#(A \cap [2^{k-1}, 2^k])$ by considering $Y_{2^k} - Y_{2^{k-1}}$ & $Y_{2^{k-1}} - Y_0$ and iterate at finer scales.



Then coupling of

- $(\tilde W_t)_{t\geq 0}$, $\tilde W_t = W_{P_t}$, P is rate 2 Poisson process, W has iid & uniform steps $\{(\pm 1, 0), (0, \pm 1)\} = \{v_1, v_2, v_3, v_4\}$
- $(Z_t)_{t\geq 0}$ is a 2D BM

$$\tilde W_t = \sum_{j=1}^4 P^j v_j, \quad P^j = \text{rate } \tfrac12 \text{ P.P.P.}$$

$$= \sum_{j=1}^4 \underbrace{(P^j - \tfrac12 t)}_{\text{couple with 1-D BM w. var } \frac12} v_j$$

constriction.

Liouville Quantum Gravity: (LQG)
Random surface with area measure $\mu = "e^{\gamma h} dA"$
surface $\simeq$ 2D Riemannian manifold.
(or rough variants).
↳ String theory, CFT, Polyakov '81
can view such surfaces as a canonical model for a random surface.

Convergence of Tutte embedding:

Recall:
M_n RPM w. disk topology.
$\phi_n : V(M_n) \to \overline{\mathbb{D}}$ Tutte embedding
$\sigma_n = 1\cdot$ vertex counting measure
$\mu_n = \phi_n * \sigma_n$ meas. in $\overline{\mathbb{D}}$.

$W^v = $ SRW on $V(M_n)$, $W^v(0) = v$, let
$\tilde\phi_n : V(M_n) \to \overline{\mathbb{D}}$ other embedding
$\tilde\phi_n(a_n) = 0$, $\tilde\phi_n(b_n) = 1$.
$\tilde\mu_n = \tilde\phi_n * \sigma_n$.

Strategy for Thm A&C:

Pick $\tilde\phi_n$ s.t.

1. $\tilde\mu_n \Rightarrow \mu$ (where μ = LQG area meas.)
2. $\|\phi_n - \tilde\phi_n\|_\infty \xrightarrow{d} 0$ ($\Rightarrow d_{prok}(\mu_n, \tilde\mu_n) \xrightarrow{d} 0$)

Q: How to pick $\tilde\phi_n$ for Thm C?

A: use Mullin bijection: (*)

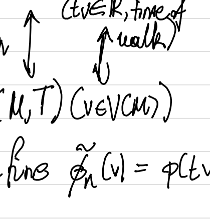
$$\begin{array}{ccc} \text{walk} & \xrightarrow[(t_v \in \mathbb{R},\ \text{time of walk})]{n\to\infty} & \text{2D BM} \quad (t \in \mathbb{R};\ \text{time for BM}) \\ \text{Mullin} \updownarrow & & \downarrow \text{Mating of trees} \\ (M,T)\ (v \in V(M)) & & \sqrt{2}\cdot\text{LQG} \quad p(t) \in \overline{\mathbb{D}} \end{array}$$

Define $\tilde\phi_n(v) = p(t_v)$.

Lemma: $\tilde\mu_n \Rightarrow \mu$
pf: ommitted (follows from definitions).

Table (*) gives first look between (M,T) & $\sqrt{2}\cdot$LQG.

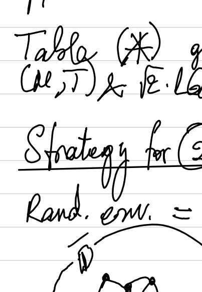
Strategy for (2): RW in random env.
Rand. env. $= \tilde\phi_n$ - emb. of M_n



$\tilde\phi_n$ - emb.
$\tilde W^n = \tilde\phi_n \circ W^{a_n}|_{[0,\tau_n]}$
RW in $\mathbb{D}$.

Two sources of randomness for $\tilde W^{a_n}$:
- the environment $\omega_n = (M_n, \tilde\phi_n)$
- the walk.

Example Session:



lattice $\mathbb{T}$: $\forall z \in \mathbb{T}$, $\mathbb{P}(z$ is black$) = p$ (site percolation).

Fact: [Kesten, Weismann] $\exists p_c \in [0,1]$ depending on lattice & dim. s.t. when
$\begin{cases} p \le p_c & \nexists \text{ infinite cluster a.s.} \\ p > p_c & \exists! \text{ infinite cluster} \end{cases}$

$p_c = 1/2$ on $\mathbb{T}$.

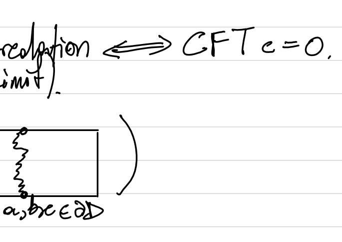
Critical percolation $\Longleftrightarrow$ CFT $c = 0$.
(scaling limit).

$\mathbb{P}\Big(\ \Big)$

$a, b, c \in \partial D$

Let $D \subset \mathbb{C}$, D_n be an approx. of D s.t. spacing $1/n$. Then define $\varphi_a^n : D \to [0,1]$

$P_a^n := \mathbb{P}\Big(\ \Big)$

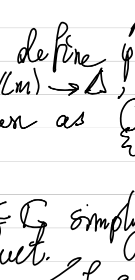


percolation interface on D_n.

Define $\overset{\text{simplex}}{\Delta}$: and $\phi_n : D \to \Delta$
$v \mapsto (p_a^n(v), p_b^n(v), p_c^n(v))$ proj.

Thm [Smirnov '01]: $\phi_n \to \phi$ as $n \to \infty$, where $\phi : D \to \Delta$.

Planar map: let m be a planar map.



for any $v \in V^{\text{black}}(m)$ sample $\sim$ Ber(1/2). Define
$\hat p_n : V(m) \to [0,1]$.
$\hat p_a(v) := \mathbb{P}\big(\exists$ percolation interface s.t. v is on same side$\big)$

and define $\hat p_b, \hat p_c$ similarly.
$\hat\phi : V(m) \to \Delta$, $v \mapsto \text{proj}_\Delta(\hat p_a(v), \hat p_b(v), \hat p_c(v))$
known as Cardy-Smirnov embedding.
↳ Oxford physicist.

$D \subseteq \mathbb{C}$ simply connected, with inner product.
$\langle f, g \rangle_D := \int_D \nabla f \cdot \nabla g \, dx$
and consider $H_0(D)$ = completion of $C_c^\infty(D)$ wrt $\langle \cdot, \cdot \rangle_D$. $\exists$ orthonormal basis of $H_0(D)$.

Sample $(\alpha_j)_{j \in \mathbb{N}}$ iid $\sim N(0,1)$.

Fact: $\sum_{j \in \mathbb{N}} \alpha_j f_j \overset{d}{=} h$ (zero bdry GFF)
↳ converges in all $H^{-\varepsilon}(D)$, $\varepsilon > 0$.

Have decomposition:

$$H_0(D) = H_0(U) \oplus \text{Harm}(U)$$

↳ f^ns on D harmonic on U.

$\Rightarrow$ restriction of GFF on two parts above are indep.

"viewed as a f^n harmonic in U."

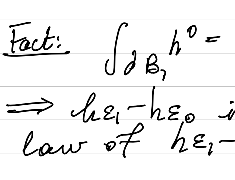
In other words, conditional on $h|_{\partial U}$, the conditional law of $h - h|_{\partial U} = h_0^U$

Claim: let $h_\varepsilon = h$ tested on leb on $\partial B_\varepsilon(z)$ prob. meas. prop. to

Then $\sqrt{2\pi}\, h_{e^{-t}}(z)$ is a BM from B_{t_0} for $t > t_0$. ($e^{-t} := \varepsilon$)

proof: let $B_t = \sqrt{2\pi}\, h_{e^{-t}}(z)$ suffices to show
1. B_t cont. sample paths ⟵ technical (Kolmogorov-Centsov type)
2. Stat. indep. increments ⟵ Domain Markov Property (DMP)
3. Correct variance ⟵ exercise (computation)

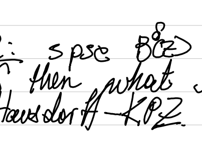
DMP:



Recall: $H_0(D) = H(B_0) \oplus \text{Harm}(B_0)$
$\Rightarrow$ Conditional on $\text{proj}_{\text{Harm}(B_0)}(h) = h^0$
$h - h^0 \perp\!\!\!\perp h^0$.

Fact: $\int_{\partial B_1} h^0 = \int_{\partial B_0} h^0$ (harmonic $\Rightarrow$ can deform contour)
$\Rightarrow h_{\varepsilon_1} - h_{\varepsilon_0}$ is indep from h^0.
law of $h_{\varepsilon_1} - h_{\varepsilon_0}$ depends only on $\varepsilon_1/\varepsilon_0$.

LQG and KPZ:



A "fractal set"!

1. Define a quantum fractal dimension.
2. Establish a relation between $d^{\text{Euclidean}}$, d^{quantum}

$\mu^\gamma \longrightarrow$ GMC measure
$e^{\gamma h(z)} dz$

$\mathbb{E}[\mu^\gamma(B_r(z))] = \pi r^2$ (exp. mart. argument)

What about? $\mathbb{E}[\mu^\gamma(B_r(z))^q]$ Minkowski-KPZ [Aru '15]

Q: spse $B_r^\delta(z)$ has a quantum area δ then what is the law of radius? Hausdorff-KPZ [RV11, DS11].

Theorem: $\gamma \in (0, \sqrt{2d})$ $B_r(z) \subseteq D$ uniformly over all such balls $\forall q < \frac{2d}{\gamma^2}$

$$\mathbb{E}[\mu^\gamma(B_r(z))^q] \asymp r^{(\frac{d}{2} + \frac{\gamma^2}{4})q - \frac{\gamma^2}{4}q^2}$$

Multi-fractal spectrum. Why multi-fractal?
($\mathbb{E}[|B_t|^q] \asymp t^{q/2}$
↳ BM

2^{-n}
A

$$M_\delta(A, 2^{-n}) = \sum_{i \in S_n} (\mathbb{1}_{S_i \cap A \neq \emptyset}) \cdot \text{Leb}(S_i)^\delta.$$

Define $d_M(A) := \inf\{\delta : \limsup_{n\to\infty} M_\delta(A, 2^{-n}) < \infty\}$.

Now quantum version:

$$M_\delta^\gamma(A, 2^{-n}) = \sum_{i \in S_n} \mathbb{1}_{S_i \cap A \neq \emptyset} \mathbb{E}[\mu^\gamma(S_i)^\delta]$$

and $q_M(A) :=$

For $d_M \in (0,1)$ set q_M s.t.
$d_M = (1 + \gamma^2/4) q_M - \frac{\gamma^2}{4} q_M^2 \Rightarrow q_M \in (0,1)$.

Then $\limsup_{n\to\infty} \mu^\gamma_{q_M}(A, 2^{-n}) = \infty$.

$\Longleftrightarrow \limsup_{n\to\infty} M_{d_M}(A; 2^{-n}) = \infty$

$\Rightarrow d_M = (1 + \gamma^2/4) q_M - \frac{\gamma^2}{4} q_M^2$ (KPZ)