Supercritical GMC

Today: guess critical value of param. (γ) in model + some intuition.
Based on work with F. Bertacco.

Cov f^n: $\mathbb{E} X_x X_y \sim -\log|x-y|$, $x,y \in \mathbb{R}^d$. (+ $f(x,y)$ "smooth", i.e. Hölder)
$\Rightarrow$ can't be realized as rand. f^n, but can as rand. d^n!

Test f^n: $\mathbb{E} X(\varphi)X(\psi) \sim \iint \varphi(x)\psi(y)\log|x-y|\,dx\,dy$
log not quite pos. definite. (✗)

Interested in rand. meas. (I) $M_\gamma(dx)$ "=" $C\, e^{\gamma X(x)}dx$ and "exponentiate" (X dist.)
C renorm const.

Q: what's multiplicative about GMC?
Q: what could we mean by (✗) $\to$ regularise and "renormalise", taking limit, show indep. of approx. mollification.

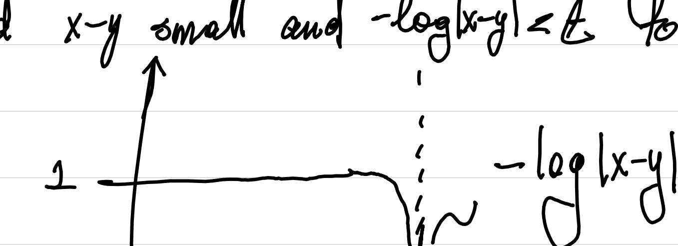
Take $\Xi : \mathbb{R}^d \to \mathbb{R}$, $\Xi \in C_c^\infty(\mathbb{R}^d)$, $\int \Xi^2 = 1$
ξ a Space-time white noise, i.e. Gauss. dist. on $\mathbb{R}^{d+1}$
$\mathbb{E}\,\xi(\varphi)\xi(\psi) = \langle \varphi, \psi\rangle$, φ,ψ white noise.
"Disjoint regions in space time $\Rightarrow$ field's indep."
and field $X_t(x) = \int_{\mathbb{R}^d}\int_0^t \Xi(e^r(x-y))\, e^{\frac{dr}{2}}\,\xi(dy\,dr)$
$:= \xi\big(\mathbb{1}_{[0,t]}\,\Xi(e^{\cdot}(x-y))\big)$
well-def^n for any test fn $\in L^2(\mathbb{R}^{d+1})$.

Observe, X_t (smooth in x, for fixed t) and for fixed x, $X_t(x)$ BM.

Now, $\mathbb{E}\,X_t(x)X_s(y) = \int_{\mathbb{R}^d}\int_0^{s\wedge t} \Xi(e^r(x-\bar y))\,\Xi(e^r(y-\bar y))\,e^{dr}\,d\bar y\,dr$
$= \int_0^{s\wedge t} K(e^r(x-y))\,dr$

$(K(x-y) = \int \Xi(x-\bar y)\,\Xi(y-\bar y)\,d\bar y)$.

Now fix $t \gg 1$, claim: $X_t(\cdot) \sim$ log-cov. field [provided $x-y$ small and $-\log|x-y| \le t$ (to see cutoff)] (*)



So have natural length scale

$-\log|x-y| \sim t \iff |x-y| \sim e^{-t}$

So at scale $\geq e^{-t} \rightsquigarrow X_t(\cdot)$, ($t \gg 1$), looks like log. cov field, and $\longleftarrow$ " $\leq e^{-t}$
$X_t(\cdot)$ essentially constant.

[$t \to$ reg. param. $\sim$ time]

Increments: indep. & almost iid (up to rescaling)

$X_{s,t} := X_t - X_s$
$X_{s,t}(e^{-s}\,\cdot) \overset{(d)}{=} X_{t-s}$. (‡)

Consider X_1. Let $X_1^{(k)}, k \geq 1$ be an iid. seq. of fields $\overset{law}{=} X_1$.

$X_m \overset{law}{=} \sum_{k=1}^m X_1^{(k)}(e^k\,\cdot)$ (follows from (‡)).
"squash down"

So multiplicative comes from rescaling "increments".

Now, want to replace in (I) $X \longmapsto X_t$ a "regularised version of X".

and set $\mu_\gamma^t(dx) = e^{\gamma X_t(x) - \frac{\gamma^2 t}{2}}dx$
norm. so $\mathbb{E}[\mu_\gamma(\cdot)] = \mathrm{Leb}(\cdot)$.

$\mu_\gamma^t =$ non-neg. MG over space of measures
(or $\langle \mu_\gamma^t, f\rangle = MG$ in usual sense).

Want UI (better control) to establish non-trivial limit.

Let $A \subseteq \mathbb{R}^d$, w. $\lambda(A) < \infty$ ($\lambda :=$ Leb). Then

$\mathbb{E}\big[(\mu_\gamma^{(t)}(A))^2\big] = \iint_{A\times A} \mathbb{E}\, e^{\gamma(X_t(x)+X_t(y)) - \gamma^2 t}\,dx\,dy$

$\quad \mathbb{E}\,e^{\gamma(X_t(x)+X_t(y))} = e^{\gamma^2/2\,\mathbb{E}[(X_t(x)+X_t(y))^2]}$
$\quad \approx e^{\gamma^2 t - \gamma^2\log|x-y| + O(1)}$

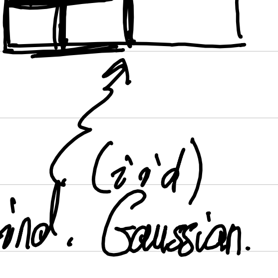
$\approx \iint_{A\times A} \frac{1}{|x-y|^{\gamma^2}}\,dx\,dy$ (reg. version at scale t)

$< \infty$ if $\boxed{\gamma^2 < d}$ $\rightsquigarrow$ Overkill (would suffice to control L^p, $p \in (1,2)$)

Q: why this kind of reg.? Very convenient to have MG structure.

Turns out, crit value: $\underline{\underline{\gamma^2 = 2d}}$

Toy model: discretise X_1 "at all scales". Work with dyadic scales,
Can take $X_{\log 2}^{(k)} \Rightarrow X_{m\log 2} \overset{law}{=} \sum_{k=1}^m X_{\log 2}^{(k)}(2^k\,\cdot)$



At any scale, in each box place ind. (iid) Gaussian.

Let $c = 2^d$. At level n, have c^n boxes.
Index, $\xi^{(n)}_{x_1,\dots,x_n}$, $x_i \in \{1,\dots,c\}$ and let
$W^{(n)}_{x_1,\dots,x_n} = e^{\gamma\xi^{(n)} - \gamma^2/2\,\log 2}$ $\mathbb{E}(\cdot\cdot) = 1$.
$\xi^{(n-1)}_{x_1,\dots,x_{n-1}}$, $\Rightarrow x_1,\dots,x_n$ "sits above" $x_1,\dots,x_{n-1}$.
(Var of ξ's $= \log 2$).
and replace Euclidean distance with "hierarchical" distance (not too different).

Define $Y_m = c^{-m}\sum_{x_1,\dots,x_m}\prod_{k\leq m} W^{(k)}_{x_1,\dots,x_k}$ (✱✱)
c^{-m}: "$\sim$ Leb. m. of small box".

Want: obtain a bound on $\mathbb{E}\,Y_m^p$ uniform in m, for some $2 > p > 1$.

Can rewrite (at least in law):

$Y_m = \frac{1}{c}\sum_{x_1=1}^{c} W_{x_1}^{(1)}\cdot Y_{m-1}^{(x_1)}$ $\to$ iid copies of Y_{m-1} & $⫫\, W^{(1)}_{x_1}$.

$\mathbb{E}\,Y_{m+1}^p = \mathbb{E}\Big(\frac{1}{c}\sum_{k=1}^{c} W_k Y_{m,k}\Big)^{p/2\cdot 2}$ ($p/2 < 1 \Rightarrow (\cdot)^{p/2}$ concave $\Rightarrow$ sub. additive)

$\leq \frac{1}{c^p}\cdot\mathbb{E}\Big(\sum_{k=1}^{c} W_k^{p/2}\cdot Y_{m,k}^{p/2}\Big)^2$

$= \frac{1}{c^p}\Big(c\cdot\mathbb{E}\,W^p\cdot\mathbb{E}\,Y_m^p + \frac{c\cdot(c-1)}{2}(\mathbb{E}\,W^{p/2})(\mathbb{E}\,Y_m^{p/2})^2\Big)$
diagonal terms + ind.; ≤ 1, Jens.; ≤ 1 by Jensen; off-diagonal terms.

$\Rightarrow$ Condition for UI: $\mathbb{E}\,W^p \leq c^{p-1}$.
(discrete Gronwall/recurrence argument).

$W \simeq e^{\gamma\sqrt{\log 2}\,Z - \frac{\gamma^2}{2}\log 2}$
So $W^p = e^{p\gamma\sqrt{\log 2}\,Z - \gamma^2/2\,\log 2\,p}$
So $\mathbb{E}\,W^p = 2^{\gamma^2/2\,(p-1)p}$.
So need $\gamma^2 p < 2d$. (Subcritical regime)

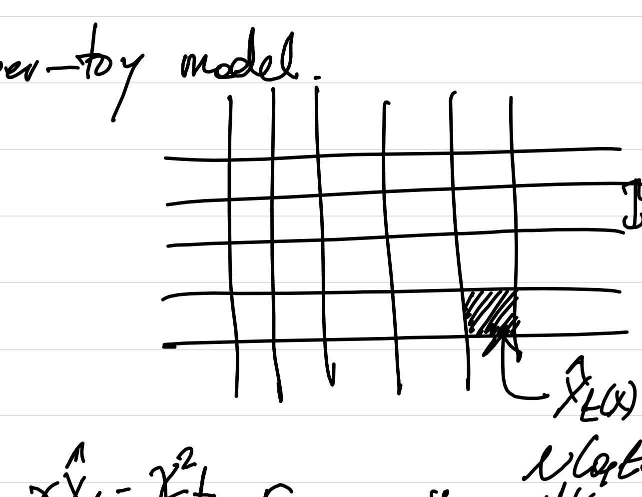
Have conv in prob. so can say
$\mu_\gamma(dx) = e^{\gamma X(x)}dx$ for some random field.

How does conv. happen in supercritical regime?

Gaussian fields $X_t(x) \xrightarrow{law} X(x)$, $x \in \mathbb{R}^d$
$\mathbb{E}\, X(x)X(y) = -\log|x-y| + O(1)$

Tried to give meaning to $e^{\gamma X_t - \frac{\gamma^2}{2}t} \to : e^{\gamma X}$: "Wick product".
"Worked" for $\gamma^2 < 2d$.

$\gamma^2 > 2d$: Super-toy model.



Look at $e^{\gamma X_t - \frac{\gamma^2}{2}t}$. For γ "small" ($\gamma^2 < 2d$) expect some LLN. $\to$ Leb

$$P(X_t(x) > \alpha t) = P(\sqrt{t}\,N(0,1) > \alpha t) = P(N(0,1) > \alpha\sqrt{t}) \sim \frac{e^{-\frac{\alpha^2}{2}t}}{\sqrt{t}} \quad (*)$$

Contribution from "exceptional" boxes s.t. $X_t(x) \approx \alpha t$:
$e^{-\frac{\alpha^2 t}{2} + \alpha\gamma t - \frac{\gamma^2}{2}t}$ — fraction of boxes
$= e^{-\frac{(\alpha-\gamma)^2}{2}t}$. If $\alpha \neq \gamma$ ⇒ small cont.
When $\alpha = \gamma$ have $\sim O(1)$ cont. (modulo neglected poly. terms).

Limiting meas. ⊥ wrt Leb. meas. on γ-thick pts:
$$A_\gamma = \{x \in \mathbb{R}^d : \lim_{t\to\infty} \frac{X_t(x)}{t} = \gamma\}$$ (Hausdorff dim. ...)

Q: largest value of $X_t(x)$?
$$P(X_t(x) > k) = P(N > \tfrac{k}{\sqrt{t}}) \sim \frac{\sqrt{t}}{k} e^{-\frac{k^2}{2t}} \overset{want}{\sim} e^{-dt}$$
$\Rightarrow -\frac{k^2}{2t} + \frac{1}{2}\log t - \log k = -dt$
$\Rightarrow -t\log t + 2t\log k + k^2 = 2dt^2$

Ansatz: $k = \sqrt{2d}\,t - S$
$\Rightarrow \log t - 2S\sqrt{2d} = 0$
$\Rightarrow S = \frac{\log t}{2\sqrt{2d}} + O(1)$.

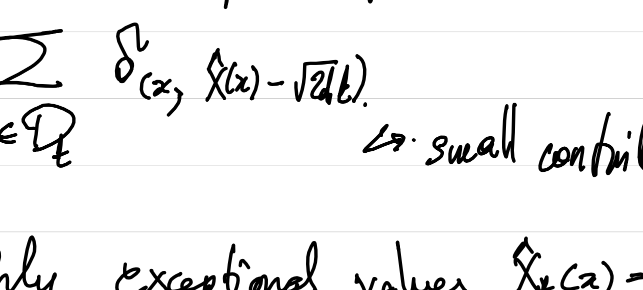
$\Rightarrow$ (#) $K = \sqrt{2d}\,t - \frac{3\log t}{2\sqrt{2d}} + O(1) \xrightarrow{law} G(x)$ — in log. corr. case ... maximum

So for γ large, should do different normalisation, get localisation, limit collection of Dirac meas.

$\gamma^2 > 2d$: $e^{\gamma X_t(x) + (d - \gamma\sqrt{2d})t}$

$O(1)$ boxes contribute (#)

(**) Gaussian dist.



$$\to e^{-\frac{(\sqrt{2d}\,t+\delta)^2}{2t}} = e^{-2dt - \sqrt{2d}\,\delta + O(\frac{\delta^2}{t})}$$

"Out in the tail, dist $\sim$ Exp($\sqrt{2d}$)". So, tells us contributions of each box and view as point process on $\mathbb{R}_+$":
Consider the point process on $\mathbb{R}^d \times \mathbb{R}_+$ given by

$$(\dagger) \sum_{x \in D_t} \delta_{(x,\, X(x) - \sqrt{2d}\,t)}$$
small contributions $\to -\infty$ (don't see them)

Only exceptional values $X_t(x) = \sqrt{2d} + O(\delta)$.
So expect $(\dagger) \Longrightarrow$ P.P.P.$(dx \otimes e^{-\sqrt{2d}\,\delta}\, d\delta)$.

So $\int f(x)\, e^{\gamma X(x) + (d - \gamma\sqrt{2d})t}\, dx \Rightarrow \sum_{(x,s)\in \text{P.P.P.}(\dots)} f(x)\, e^{\gamma s}$ (=) — test f^n — (⋆)

let $m = e^{\gamma s}$ $dm = \gamma e^{\gamma s} ds$.
So $e^{-\sqrt{2d}\,s - \gamma s}\, dm = \gamma e^{-\sqrt{2d}\,s}\, ds$
$= m^{-1-\beta} dm$, $\beta = \frac{\sqrt{2d}}{\gamma} \in (0,1)$.

$$\Rightarrow (=) \sum_{(x,m)\in \text{P.P.P.}(dx \otimes m^{-1-\beta} dm)} m \cdot f(x)$$
not intgble at zero. PPP has inf. many points near zero (going to accumulate). But quantity above is well-defined due to m.

$\gamma = \sqrt{2d}$ ⇒ not intgble but have a lot of mass from bulk → still conv. to Leb. so just change normalisation to $t^{1/2} e^{\gamma X_t - \frac{\gamma^2}{2}t}$.

Q: recover spatial structure of fields $X_t(x)$ in lim, other than Leb?

recall $X_t = X_s + X_{s,t} \overset{law}{=} X_s + \underbrace{X_{t-s}(e^s \cdot)}$
meas. (⋆) to scale invariant.
But — should conv. to something (s fixed. $t \gg 1$)
but the X_s term → conv. to non-unif. P.P.P. contradiction, so will
So hope to conv. to (⋆)

Call. P.P.P.$_\beta(dx)$
$$\left[\; \nu \sim \text{P.P.P.}_\beta(\mu) \quad \int f(x)\,\nu(dx) = \sum_{(x,m)\in \text{P.P.P.}(\mu \otimes m^{-1-\beta} dm)} m f(x) \;\right]$$

Claim: $\underbrace{X_{t-s}(e^s \cdot)}$ for $s \gg 1$, $t \gg s$ $\longrightarrow$ PPP$_\beta(dx)$ and put correct s,t-dep. prefactors to μ_t^γ ...

Now, $e^{\gamma X_t} \simeq e^{\gamma X_s} \cdot \text{PPP}_\beta(dx)$
keep track of exp. normalisations log terms more tricky.

Recall: $\nu \sim \text{PPP}(\mu)$.
$$\mathbb{E} e^{-\int f(x)\nu(dx)} \text{ "=" } \prod_x \underbrace{\mathbb{E} e^{-f(x)\nu(dx)}}_{e^{-\mu(dx)(1-e^{-f(x)})}}$$
$$\vdots\; = e^{-\int (1-e^{-f(x)})\mu(dx)}$$

So for $\nu \sim \text{PPP}_\beta(\mu)$
$$\mathbb{E} e^{-\int f(x)\nu(dx)} (=) \exp\left(-\int (1-e^{-mf(x)})\, m^{-1-\beta}\, dm\, \mu(dx)\right)$$

Let $q = mf(x)$
$dq = dm\, f(x)$
$\frac{dq}{q} = \frac{dm}{m}$

$$(=) \exp\Big(-\int_{\mathbb{R}^d}\underbrace{\int_0^\infty (1-e^{-q})\, q^{-1-\beta} dq}_{c_\beta}\, f^\beta(x)\,\mu(dx)\Big).$$
$$= \exp\Big(-c_\beta \int_{\mathbb{R}^d} f^\beta(x)\,\mu(dx)\Big)$$

So $e^{\gamma X_s}\, \text{PPP}_\beta(dx) = \text{PPP}_\beta(e^{\beta\gamma X_s(x)} dx)$.
$(=) \text{PPP}_\beta(e^{\sqrt{2d}\, X_s(x)} dx)$.

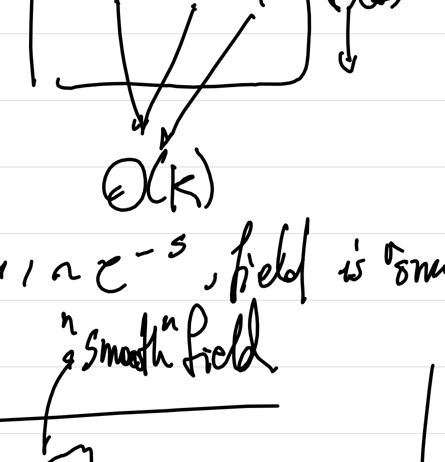
Suggests normalise X_t ⇒ conv. to non-deg. limit ⟹ rand. purely atomic measure with rand. intensity given in terms of critical GMC. Two layers of randomness become indep!

$(=) \longrightarrow \text{PPP}_\beta(\text{GMC}_{\sqrt{2d}})$

Conj. Duplantier, Sheffield & Vargas.
$m^{-1-\beta}$ power law for masses "known" by physicists before.

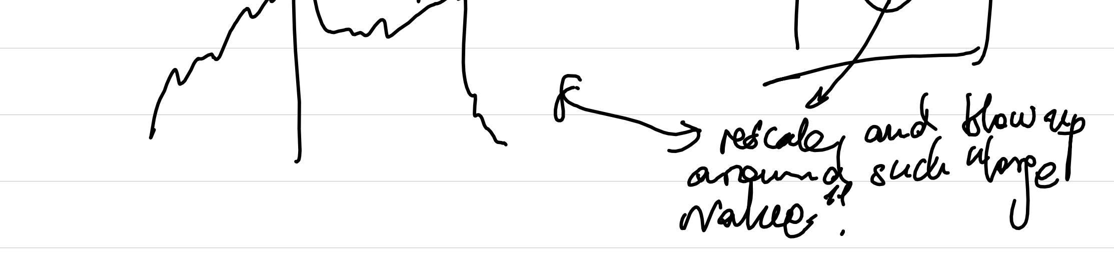
With $\mu_t^\gamma(dx) := t^{(\dots)} e^{\gamma X_t(x) - (\dots)t} dx$, look at map $s \longmapsto \mu_{t+s,\gamma}$ (measure-valued).
how do masses of limiting measure changes (expect locations to be the same).

So expect:



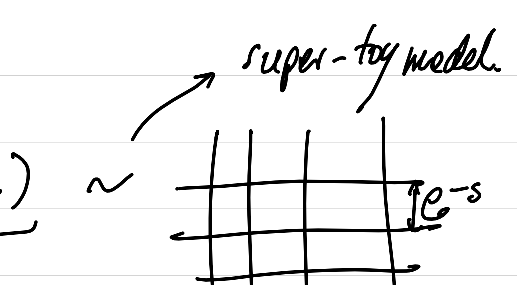
at small scale, $\sim e^{-s}$, field is "smooth".

Expect:



$e^{\gamma X_t + (d-\sqrt{2d}\gamma)t}$

$X_t = X_s + \tilde{X}_{t-s}(e^s\cdot) \sim$



max in box $\sim \sqrt{2d}(t-s)$ in each box $+ \mathrm{Exp}(\sqrt{2d})$ (of order)

So global max de unit order domain $\sim$ (of order) $\sqrt{2d}(t-s) + \sqrt{\frac{d}{2}}\, s$

(Since $\mathrm{Exp}(-\sqrt{2d}\,k) \sim e^{-ds}$ for $k \sim \frac{s \cdot d}{\sqrt{2d}}$)

"Width" of the peak around the global max $\sim e^{-t}$ ($\tilde{X}_{t-s}$ has peaks of width $O \sim e^{-(t-s)}$ & rescale by e^s)

So concentration of max. peaks $\sim \exp(-ds + \gamma\sqrt{2d}(t-s) + \gamma\sqrt{\frac{d}{2}}\, s)$.

$\left(\int_A e^{\gamma \tilde{X}_{t-s}(t-s)} dx\right)$

Now, $e^{\gamma X_t + (d-\sqrt{2d}\gamma)t} = \underbrace{e^{(d-\sqrt{2d}\gamma)t + \gamma\sqrt{\frac{d}{2}}s}}_{\text{"Correct normalisation"}} e^{\gamma X_{s,t}} \left(e^{\sqrt{2d}X_s - sd}\right)^{\gamma/\beta}$, $\beta = \frac{\sqrt{2d}}{\gamma}$

$\Rightarrow \mathrm{PPP}(\mathrm{Leb.})$

"correct norm" $\Rightarrow \mathrm{GMC}_{\sqrt{2d}}$

Now can convolve with mollifier at scale $\varepsilon = e^{-t}$

$X_{(\varepsilon)} = X * \rho_\varepsilon$, $\rho_\varepsilon(x) = \varepsilon^{-d}\rho(x/\varepsilon)$, $\int \rho = 1$

and recover ~~some~~ limit. For large class of mollifiers:

$X_{(\varepsilon)} \overset{\text{law}}{=} X_t + W(e^t\cdot) + O(\varepsilon)$ ($\varepsilon = e^{-t}$) in sup.

where W is a stationary, smooth Gaussian field with decaying covariance.

Intuition about local maxima & glob. max >> loc. maxima (by indep.)

At $O(1)$ scales, properties of max don't change due to decay in cov.

Let $\nu \sim \mathrm{PPP}(m^{-1-\beta}dm)$, $(Z_i)_i$ iid

$\nu = \sum \delta_{m_i}$, $\tilde{\nu} := \sum \delta_{m_i Z_i}$

$\Rightarrow \tilde{\nu} \sim \mathrm{PPP}(c\, m^{-1-\beta}dm)$ (Compute Laplace transform)

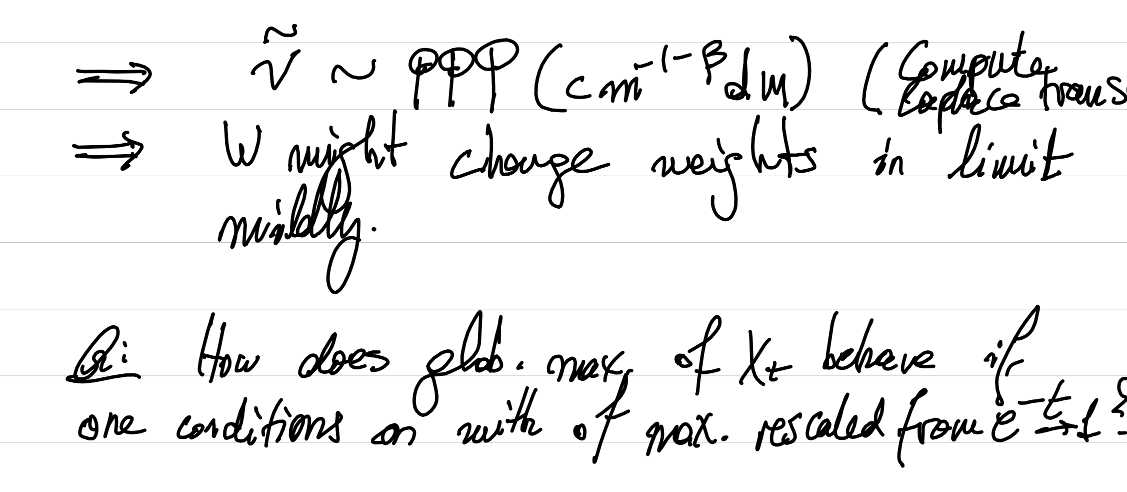
$\Rightarrow$ W might change weights in limit mildly.

Q: How does glob. max of X_t behave if one conditions on width of max. rescaled from e^{-t} to 1?

Let $\bar{X}_t = X_t(e^{-t}\cdot) \to O(1)$ correlations

Look at $\bar{X}_t - \bar{X}_t(0)$, conditioned on $\bar{X}_t(0) \sim \sqrt{2d}\,t$

"Change coordinates"



$X_t = \int_{\mathbb{R}^d}\int_0^t K(e^r(x-y))e^{\frac{rd}{2}}\xi(dy,dr)$

choose h_t so that terms become $\perp$

$\bar{X}_t(x) = \bar{X}_t(x) - h_t(x)\bar{X}_t(0) + h_t(x)\bar{X}_t(0)$

So computing covariance:

$\mathbb{E}\left[\bar{X}_t(0)\bar{X}_t(x)h_t(x) - t\,h_t^2(x)\right] = 0$

So can take $h_t(x) = \frac{1}{t}\int_0^t K(e^{-s}x)\,ds$

Have decomposition of $\bar{X}_t$

$\bar{X}_t(x) = \int_0^t K(e^{r-t}x)\, d\bar{X}_r(0) + Z_t(x)$ — BM, t indep.

(at the level of processes)

$\mathbb{E}Z_t(x)Z_t(y) = \int_0^t \left(K(e^{-r}(x-y)) - K(e^{-r}x)K(e^{-r}y)\right)dr$

Now $\bar{X}_t(x) - \bar{X}_t(0)$ cond. on $\bar{X}_t(0) \sim \sqrt{2d}\,t$ (afterward $\bar{X}(0) \approx B$ with drift)

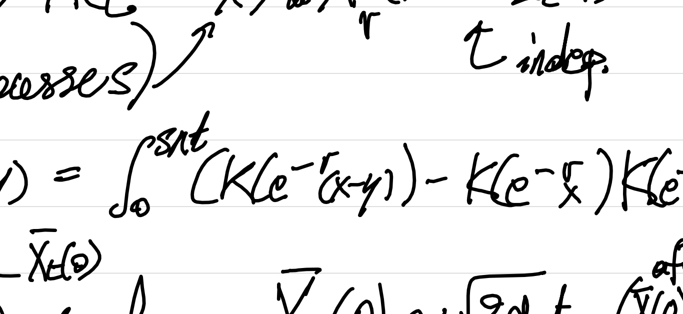
$\Longleftrightarrow \sqrt{2d}\int_0^t (K(e^{r-t}x)-1)\,dr + \int_0^t (K(e^{-r}x)-1)\,dB + Z_t(x)$

as $t\to\infty$

$\to \sqrt{2d}\int_0^\infty (K(e^{-r}x)-1)\,dr + \int_0^\infty (K(e^{-r}x)-1)\,dB + Z_\infty(x)$.

$a(x) \sim -\log|x|$, x large

Intuition:



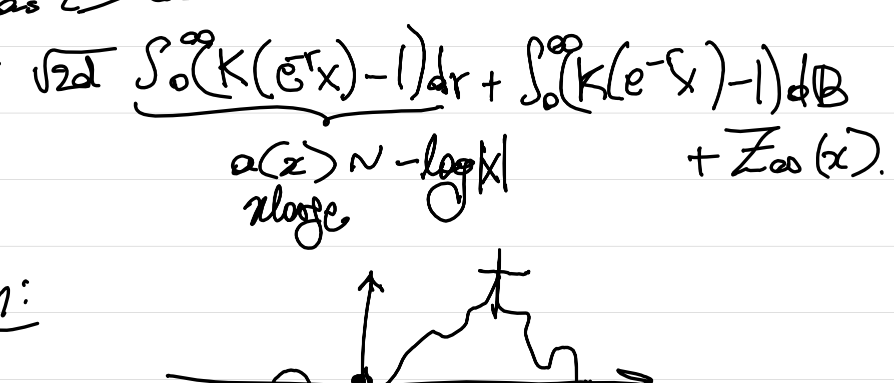
turns out field above does not have a global max. Why?: $Z_t(x) = \int_{\mathbb{R}^d}\int_0^t \left(K(e^{-r}(x-y)) - K(e^{-r}x)K(e^{-r}y)\right)e^{-\frac{rd}{2}}\xi(dy,dr)$

For $x,y \sim e^b \Rightarrow Z_\infty \sim Z_b(x) \approx X_b(x)$ (constant)

Expect: $\sup_{|x|\le e^b} Z_\infty(x) \sim \sqrt{2d}\, b \sim \sqrt{2d}\log|x|$

"Chop space into exp. annuli" $\Rightarrow \exists$ eps. at where $Z_\infty(x)$ cancels diverging drift $a(x)$.

So chance ...

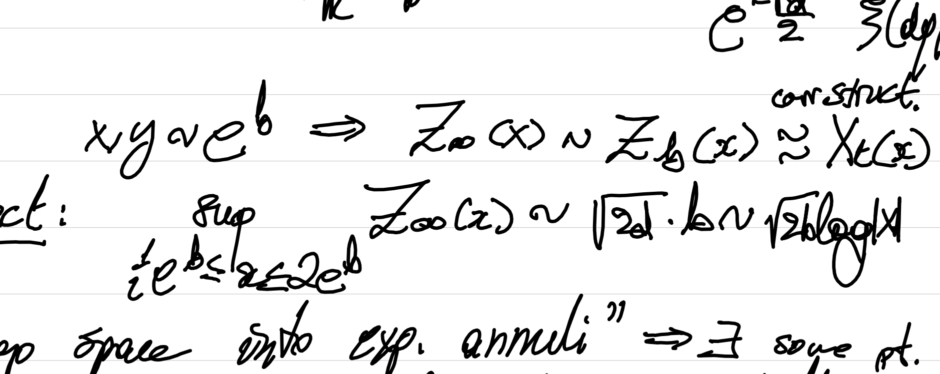


and $\int_0^\infty (K(e^{-s}x)-1)\,dB \sim -B_{\log|x|}$.

Need to further condition on origin being close to origin.

Condition process to be $\le \lambda$ on growing balls $\Rightarrow$ get random field (see paper).

Condition field to stay below more general barriers.



Cond. BM $\sim \mathrm{Bes}(3)(x)$ & really do have localised global max. and then change coordinates.

Want/hope: Law of limiting field indep. of barrier Y.

However, it does depend on λ.

Call Ψ := field near local maxima

"location" "max" "shape" $\propto \mathrm{PPP}(e^{-Fdx}\,ds \otimes dx \otimes P(d\Psi))$ (loc. maxima)

write $P :=$ law Ψ

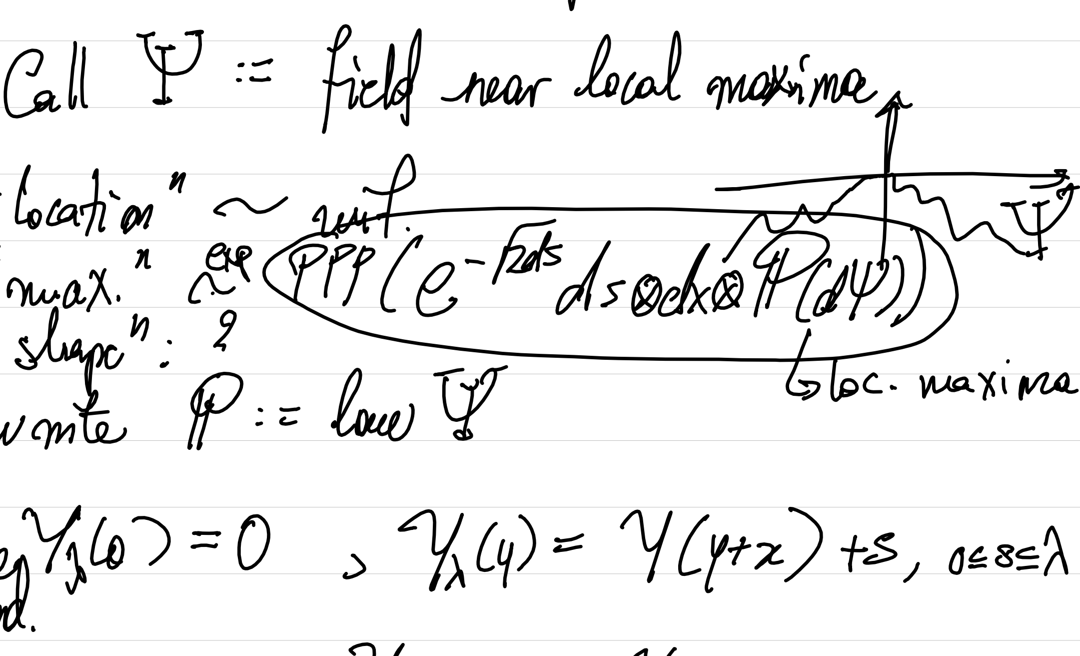
Having $Y_\lambda(0) = 0$, $Y_\lambda(y) = Y(y+z) + s$, $0 \le s \le \lambda$

$\Rightarrow s = -Y(z)$, $Y_\lambda(x) \le 0$.

So for $Y_\lambda(0) \sim \mathrm{PPP}(e^{\ldots})$ loc. max

So $\mathbb{E}f(Y_\lambda) \propto \mathbb{E}f(Y)\left[e^{\ldots}\right]$

Can invert formula to obtain law of Y in terms of λ and show that this agrees, is indep. of λ (see analogue for BM in ex. class).



Look at evolution in t:

what guarantees stationarity in time? Stat. increments? not true. need stat. increments modulo some shift.

$(\tau_s Z)(t) = Z(t+s) - Z(s)$

$Z_0 = 0$, $Z \in C(\mathbb{R}_+ \times \mathbb{R})$

Take $F_i : \mathbb{R} \times C_0 \to \mathbb{R}$

$\mathbb{E}e^{-\sum_i \sum_j F_i(z_j + Z_j(t_i), \tau_{t_i} Z_j)}$

$= \exp\left(-\int\left(1 - \prod e^{-F_i(x + B_{t_i}, \tau_{t_i}B)}\right)e^{-\alpha x}\,dx \otimes P(dB)\right)$

Now take $0 \le t \le$ min of s_i:

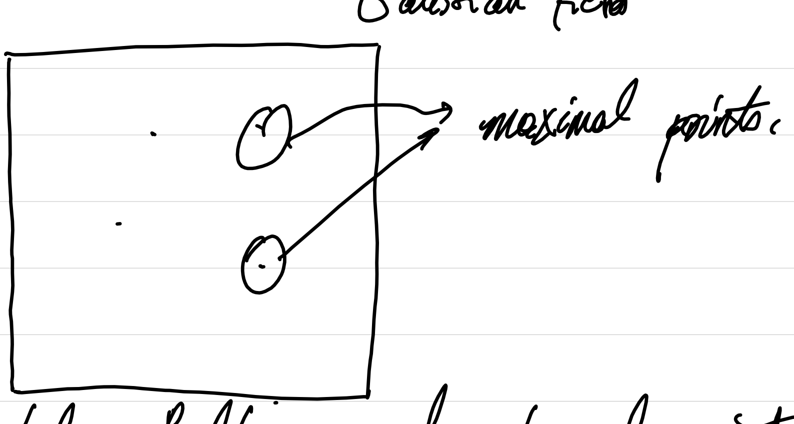
$= \exp\left(-\int\left(\ldots F_i(x + B_t + B_{t_i - t}, \tau_{t_i - t}(\tau_t B))\right)\ldots\right)$

$= \exp\left(-\int\left(\ldots F_i(x + \ldots, \tau_{t_i-t}(\tau_t B))\right)e^{-\alpha(x - \ldots)}\,dx\,P(dB)\right)$

Want: (Stationarity)

$\int F(\tau_t B)\,e^{\alpha B_t}\,P(dB) = \int F(B)\,P(dB)$

Gaussian field



<u>Goal:</u> study field around extremal points.

More specifically, talk about BM.

<u>Brownian motion around extreme points</u>

Let B be a two-sided B.M. $B_0 = 0$.

<u>Strategy:</u>

1) Take $\lambda > 0$, $R > 0$.
   Let $X_{\lambda,R}$: B conditioned on event $\{\sup_{[-R,R]} B(t) < \lambda\}$

<u>Prop 1:</u> $X_{\lambda,R} \xrightarrow{\text{law}} X_\lambda \ (R \nearrow +\infty)$
and $\lambda - X_\lambda$ is Bessel(3) process.

Want to recentre around t^*.

Define Ψ_λ s.t. , $\forall F : \mathcal{C} \longrightarrow \mathbb{R}$ $\quad \tau_t X(s) = X(t+s) - X(t)$. shift operator

$$\mathbb{E}[F(\Psi_\lambda)] \propto \mathbb{E} \frac{F(\tau_{t^*} X_\lambda)}{|\{u : X_\lambda(u) \geq X_\lambda(t^*) - \lambda\}|}$$

<u>Prop 2:</u> Ψ_λ is indep. of λ.

<u>Proof (Prop 1):</u>
Let $\tilde{B}$ be another BM, $\tilde{B}(0) = \lambda$
$\tilde{X}_{\lambda R}$: conditioned to be positive at $[-R,R]$. Choose $0 < t_1 < t_2$

Compute $\mathbb{P}(\tilde{X}_{\lambda,R}(t_2) = x_2 \mid \tilde{X}_{\lambda,R}(t_1) = x_1)$

$$= \frac{\mathbb{P}(\tilde{X}(t_2) = x_2, \tilde{X}(t_1) = x_1)}{\mathbb{P}(\tilde{X}(t_1) = x_1)}$$

$$= \frac{\mathbb{P}(\tilde{B}(t_2) = x_2, \tilde{B}(t_1) = x_1, \inf_{[-R,R]} B > 0)}{\mathbb{P}(\tilde{B}(t_1) = x_1, \inf_{[-R,R]} B > 0)}$$

$$\mathbb{P}(\tilde{B}(t_2) = x_2, \tilde{B}(t_1) = x_1, \inf_{[0,t_2]} B > 0)\, \mathbb{P}_{x_2}(\inf_{[t_2,R]} \tilde{B} > 0)$$

Let p^0 be the transition prob. of BM killed at 0, $p^0 = p^0(0,\lambda, t_1 x_1)\, p^0(t_1, x_1; t_2, x_2)$

$$\times \big(\underbrace{1 - 2\mathbb{P}_0(B(R-t_2) > x_2}_{= \mathbb{P}(|B(R-t_2)| < x_2)}\big)$$

$$\sim \frac{x_2}{\sqrt{R - t_2}}$$

As $R \to +\infty$, we get

$$\lim_{R \to \infty} \mathbb{P}(\tilde{X}_{\lambda,R}(t_2) = x_2 \mid \tilde{x}_{\lambda,R}(t_1) = x_1)$$

$$= p^0(t_1, x_1; t_2, x_2) \cdot \frac{x_2}{x_1}$$

and obtain X_λ has same finite dim. distr.s of Bessel (3) process.

<u>Prop 3:</u> (Resampling property of X_λ).
For any G, have

$$\mathbb{E}\, G(X_\lambda) = \mathbb{E} \frac{\int G(\tau_t X_\lambda) \mathbb{1}_{X_\lambda(t) \geq X_\lambda(t^*) - \lambda}\, dt}{|\{u : X_\lambda(u) \geq X_\lambda(t^*) - \lambda\}|}$$

"Sample ~~t^*~~ unif. and recentre and condition".

<u>Proof (Prop 2):</u> let $\lambda_2 < \lambda_1$.

$$\mathbb{E} F(\Psi_{\lambda_2}) = \mathbb{E}\left(\frac{F(\tau_{t^*} X_{\lambda_2})}{|\ldots|} \right)$$

$X_{\lambda_2} = X_{\lambda_1}$ conditioned to be $\leq \lambda_2$.

$$= \mathbb{E}\left[\frac{F(\tau_{t^*} X_{\lambda_1}) \mathbb{1}_{X_{\lambda_1}(t^*) \leq \lambda_2}}{|\{u : X_{\lambda_1}(u) \geq X_{\lambda_1}(t^*) - \lambda_2\}|} \right]$$

$$\overset{?}{=} \mathbb{E}\left[\frac{F(\tau_{t^*} X_{\lambda_1})}{|\{u : X_{\lambda_1}(u) \geq X_{\lambda_1}(t^*) - \lambda_1\}|} \right] = F(\Psi_{\lambda_1}).$$

$=: G$

Shift X by u uniform on $\{u : X_{\lambda_1}(u) \geq X_{\lambda_1}(t^*) - \lambda_2\}$ indep. of X_λ

$\mathbb{1}(X_{\lambda_1}(t^*) - X_{\lambda_1}(t) \leq \lambda_2)$.