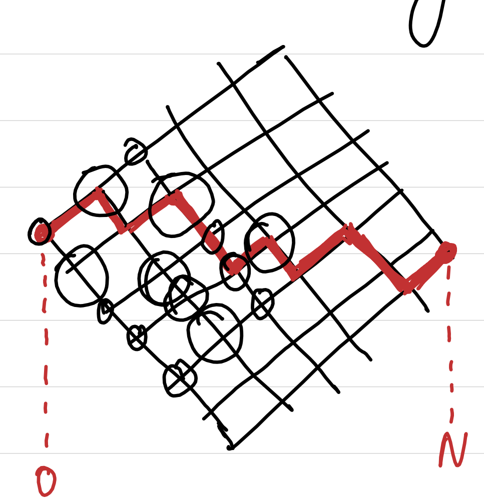
The Localisation Transition For Directed Polymers ($d \geq 3$)



Simple Random Walk (SRW) on $\mathbb{Z}^d$

$N \geq 1$ $\Omega_N := \{ (X_n)_{n=0}^N \in (\mathbb{Z}^d)^{N+1} : X_0 = 0$ and $\forall n \in [\![1, N]\!], \|X_n - X_{n-1}\|_{\ell_1} = 1 \}$

$\Updownarrow$

$X_n \sim X_{n-1}$.

SRW P_N is the uniform probability on Ω_N $|\Omega_N| = (2d)^N$.

$(n, X_n^N)_{n=0}^N \in \mathbb{Z}_+ \times \mathbb{Z}^d$ (directed SRW).

Two important results:

① SRW endpoint distr^n is delocalised $\mu_N := P_N(X_N \in \cdot)$, i.e.

$$\max_{x \in \mathbb{Z}^d} \mu_N(x) \asymp N^{-d/2} \Rightarrow \lim_{N \to \infty} \|\mu_N\|_\infty = 0.$$

② It's diffusive, $N \gg 1 \Rightarrow |X_N| \asymp N^{1/2}$

Extend X to $\mathbb{R}_+$ $(X_t)_{t \geq 0}$ as follows:

$X_t := (1-m) X_m + (m+1-t) X_m$, if $t \in [m, m+1]$.

$(X_t^{(N)})_{t \in [0,1]} = \left(\frac{X_{tN}}{\sqrt{N}} \right)_{t \in [0,1]}$.

DONSKER'S THEOREM

$(X_t^{(N)})_{t \in [0,1]} \xRightarrow{N \to \infty} (B_t)_{t \in [0,1]}$, where $B.$ is a Brownian Motion with covariance $\frac{1}{d} I_d$.

[$\mathcal{C}$ denotes set of cont. f^{n}s $[0,1] \to \mathbb{R}^d$ equipped with the sup. norm.]

If $\varphi : \mathcal{C} \to \mathbb{R}$ is bdd. cont. then

$$\lim_{N \to \infty} \mathbb{E}_N[\varphi(X^{(N)})] = Q(\varphi(B))$$

$\hookrightarrow$ Wiener measure.

Directed Polymer in a Random Environment

STATE SPACE: Ω_N

Environment: $\omega = (\omega_{n,x})_{n \geq 1, x \in \mathbb{Z}^d}$ i.i.d

For $X \in \Omega_N$ $H_N^\omega(X) = \sum_{n=1}^N \omega_{n, x_n}$

Define measure $P_N^{\beta,\omega}(X) = \dfrac{e^{-\beta H_N^\omega(X)}}{Z_N^{\beta,\omega}}$

$Z_N^{\beta,\omega} := \sum_{x \in \Omega_N} \exp(-\beta H_N^\omega(X))$

Parameter "β = inverse temperature"

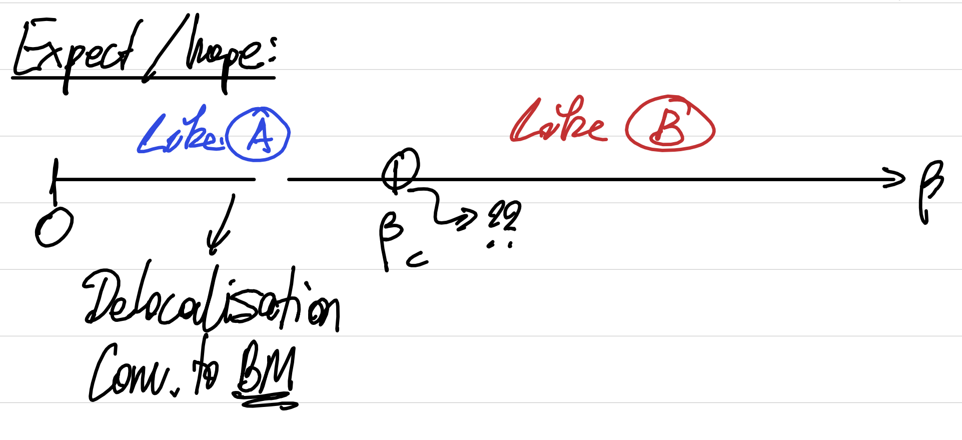
Extremal cases:

Ⓐ $\beta = 0$, $P_N^{\beta,\omega} = P_N \longrightarrow$ SRW

Ⓑ $\beta = \infty$ $(P_N^{\infty,\omega} = \lim_{\beta \to \infty} P_N^{\beta,\omega})$

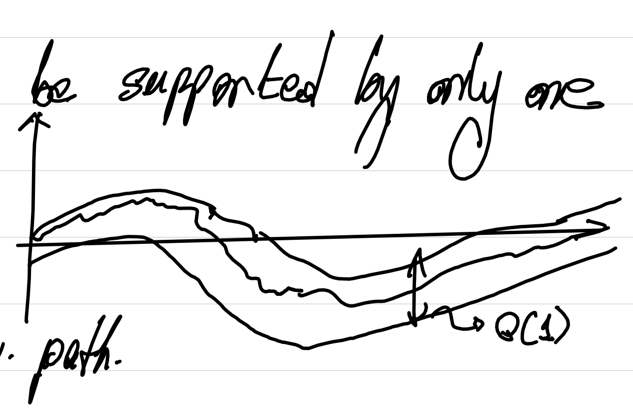
$P_N^{\infty,\omega}$ is a.s. a Dirac mass on a single path.

Expect/hope:



For $P_N^{\beta,\omega}$ cannot be supported by only one path if $\beta < \infty$.

Expect localisation in $O(1)$ of some ω-fav. path.



When $\beta > \beta_c$, do not expect: $\max_{x \in \mathbb{Z}^d} P_N^{\beta,\omega}(X_N = x) \xrightarrow{N \to \infty} 0$.

Our goal: Confirm this picture for $d \geq 3$.
(For $d = 1, 2$ it also holds, but no phase transition, i.e. $\beta_c = 0$).

assume $\forall \beta \in \mathbb{R}, \lambda(\beta) < \infty$

We study: $W_N^{\beta,\omega} = \dfrac{Z_N^{\beta,\omega}}{\mathbb{E}[Z_N^{\beta,\omega}]}$.

Compute $\mathbb{E}[Z_N^{\beta,\omega}] = \sum_{x \in \Omega_N} \mathbb{E}[e^{\beta H_N^\omega(x)}]$ (sum of iid rvs)

$= (2d)^N (\mathbb{E}[e^{\beta \omega_{x,0}}])^N$

$$W_N^{\beta,\omega} = \sum_{x \in \Omega_N} \frac{e^{\sum_{n=1}^N (\beta \omega_{n,x_n} - \lambda(\beta))}}{(2d)^N}$$

$\lambda(\beta) = \log \mathbb{E} e^{\beta \omega_{1,0}}$.

$= E_N[e^{\sum_{n=1}^N \beta \omega_{n,x_n} - \lambda(\beta)}]$.

$= E[e^{\sum_{n=1}^N \beta \omega_{n,x_n} - \lambda(\beta)}]$.

CLAIM ① Bolthausen, '89.

$(W_N^\beta)_{N \geq 2}$ is a MG, wrt $(\mathcal{F}_N)_{N \geq 2}$, where $\mathcal{F}_N := \sigma(\omega_{n,x}, x \in \mathbb{Z}^d, n \leq N)$.

Proof: $\mathbb{E}[W_{N+1} | \mathcal{F}_N] = E[\mathbb{E}[e^{\sum_{n=1}^{N+1}(\beta \omega_{n,x_n} - \lambda(\beta))} | \mathcal{F}_N]]$

$= \mathbb{E}[e^{\sum_{n=1}^N} \cdots \mathbb{E}[e^{\beta \omega_{N+1,X_{N+1}} - \lambda(\beta)} | \mathcal{F}_N]]$

$= W_N$ $(\& \mathbb{E}[W_N] = 1)$.

$(W_N^\beta)_{N \geq 0}$ is thus a (≥ 0) MG & converges a.s.

i.e. $\lim_{N \to \infty} W_N^\beta = W_\infty^\beta$ exists a.s.

Claim ②: $\mathbb{P}(W_\infty^\beta = 0) \in \{0, 1\}$

and $\mathbb{E}[W_\infty^\beta] = \mathbb{P}(W_\infty^\beta > 0)$. ($W_\infty > 0$ a.s. $\Leftrightarrow$ W_N is U.I.)

Proof: define shift operator.

$\theta_{k,z} \omega := (\omega_{n+k, x+z})_{n \geq 1, x \in \mathbb{Z}^d}$

$\Theta_{k,z} f(\omega) := f(\theta_{k,z} \omega)$.

$\hat{W}_N^\beta(x) = \mathbb{E}[e^{\sum_{n=1}^N (\beta \omega_{n,x_n} - \lambda(\beta))} \mathbb{1}_{\{X_N = x\}}]$.

$W_{N+n} = \sum_{x \in \mathbb{Z}^d} \hat{W}_N(x) \Theta_{N,x}(W_n)$ "Markov prop^{ty}".

Take $n \to \infty$: $W_\infty = \sum_{x \in \mathbb{Z}^d} \hat{W}_N(x) \Theta_{N,x}(W_\infty)$.

(sums finite + a.s. conv).

$\{W_\infty > 0\} = \{\exists x : \Theta_{N,x}(W_\infty) > 0$ and $\mathbb{P}(X_N = x) > 0\}$.

$\hookrightarrow$ is $\sigma(\omega_{n,x} : n > N)$ measurable.

N arbitrary $\Rightarrow$ by Kolmogorov 0-1 $\Rightarrow \mathbb{P}(\cdot) \in \{0,1\}$.

$\mathbb{E} W_\infty | \mathcal{F}_N] = \sum_{x \in \mathbb{Z}^d} \underset{\mathcal{F}_N\text{-meas.}}{\hat{W}_N(x)} \mathbb{E}[\underset{\perp\!\!\!\perp \mathcal{F}_N}{\Theta_{N,x} W_\infty} | \mathcal{F}_N]$

$= W_N \cdot \mathbb{E}[W_\infty]$

$N \to \infty \Rightarrow$ LHS $\longrightarrow W_\infty \Rightarrow W_\infty = W_\infty \cdot \mathbb{E}[W_\infty]$.

If $W_\infty > 0$ a.s., say that weak disorder holds and if $W_\infty = 0$ a.s., say that strong disorder holds.

Weak disorder:

$Z_N \asymp \mathbb{E}[Z_N]$, $Z_N = \sum_{x \in \Omega_N} e^{\beta H_N(X)}$ $\sim \mathbb{E} e^{\beta H_N(X)}$

(terms contribute evenly, averaging occurs).

Proposition (Comets, Yoshida '06)
There exists $\beta_c \in [0,\infty]$ such that weak disorder holds when $\beta \in [0, \beta_c)$ & strong " " " $\beta \in (\beta_c, \infty)$.

Prop: [CSY '03, CH '02] in $d = 1, 2$, $\beta_c = 0$.

Rmk: $\beta_c < \infty$ if ω has no atoms.

Prop: [B'89] $\beta_c > 0$ when $d \geq 3$

Theorem: [IS '88, B'89, ..., CSY'06]
"If weak disorder holds $\Rightarrow$ Donsker."
let $\varphi : \mathcal{C} \to \mathbb{R}$ cts, bdd. Then,

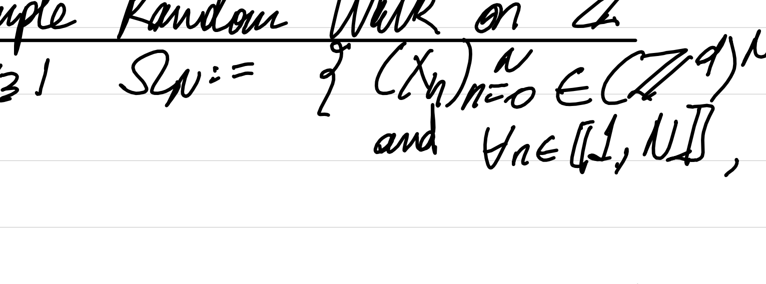
$$(*) \lim_{N \to \infty} E_N^\beta[\varphi(X^{(N)})] = Q(\varphi(B)) \text{ in prob.}$$

Rmk: holds a.s. simultaneously for all φ along a subsequence.

The free energy $F(\beta)$ is defined as

$$F(\beta) := \lim_{N \to \infty} \frac{1}{N} \log W_N^\beta \text{ a.s. (CSY '03)}$$

$F(\beta)$ is cts and non-increasing.

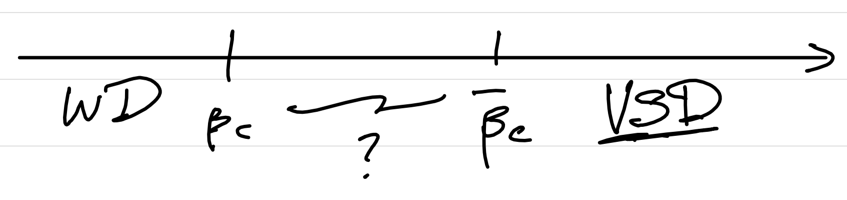


If $F(\beta) < 0$ $(\beta > \bar{\beta}_c)$, then $W_N \to 0$ exp^{y} fast in N.
Say that: Very Strong disorder.

Define $\mu_N^\beta := P_N^\beta(X_N \in \cdot)$, $I_N := \sum_{x \in \mathbb{Z}^d} (\mu_N(x))^2$.

Thm (CH'02 CSY'03)
If $F(\beta) < 0$, then $\liminf_{N \to \infty} \frac{1}{N} \sum_{n=1}^N I_n > 0$.

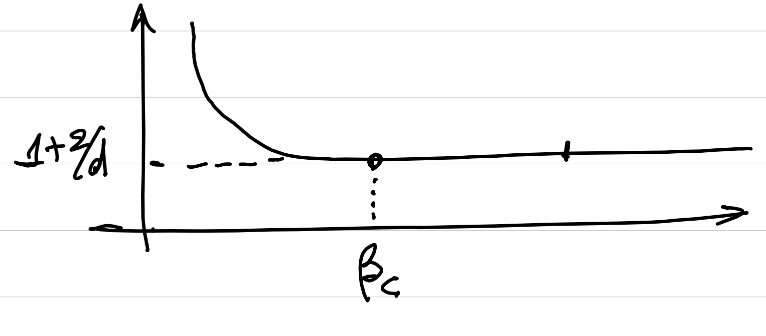
CHATERJEE BATES '20 μ_n conv. to a limit in CESARO MEAN (modulo some stuff).



THM: [JL 25+]
VERY STRONG DISORDER IS EQUIVALENT TO STRONG DISORDER
① $\beta_c = \bar{\beta}_c$
② $W_\infty^{\beta_c} > 0$ a.s.

Pf: $d \geq 3 \Rightarrow \beta_c > 0$.

Sketch: $\mathbb{E}[(W_N)^2]$ is uniformly bdd in $N \geq 1$.
$\Rightarrow \beta_c \geq \beta_2 > 0$.
For $\beta \leq \beta_c$ $\exists p^*(\beta)$ such that
$\mathbb{P}[W_\infty^\beta \geq u] \underset{u \to \infty}{\asymp} u^{-p^*(\beta)}$



WEAK DISORDER $\Rightarrow$ DIFFUSIVITY.

$$\frac{dP_N^{\beta,\omega}}{dP}(X) = \frac{e^{\sum_{n=1}^{N}(\beta\omega_{n,x_n} - \lambda(\beta))}}{W_N^\beta}$$

$$\lambda(\beta) = \log \mathbb{E} e^{\beta\omega_{1,0}}, \quad W_N^\beta = E\, e^{(\cdots)}]$$

"Weak disorder" $= W_N^\beta \xrightarrow{N\to\infty} W_\infty^\beta > 0$ a.s.

Theorem (COMETS, YOSHIDA '06)

$\mathcal{C} := C([0,1];\mathbb{R}^d)$

$X_t^{(N)} = \frac{X_{Nt}}{\sqrt{t}}$

If weak disorder holds, then $\forall \varphi \in \mathcal{C}$

$$\lim_{N\to\infty} E_N^{\beta,\omega}[\varphi(X^{(N)})] = Q(\varphi(B)), \text{ in prob.}$$

where B is a B.M. of covariance $1/d\, I_d$.

Pf. $f:\Omega \to [0,1]$ measurable.

$$E_N^{\beta,\omega}(f(X)) = \frac{E\, e^{\sum_{n=1}^{N}(\beta\omega_{n,x_n} - \lambda(\beta))} f(X)}{W_N^\beta} := \frac{W_N^\beta(f)}{W_N^\beta}$$

MG for same reason as before

$$0 \le W_N(f) \le W_N, \quad \xRightarrow{\text{U.I.}} \begin{cases} W_N(f) \xrightarrow{N\to\infty} W_\infty(f) \text{ a.s.} \\ \mathbb{E}[W_\infty(f)] = E[f(X)] \end{cases}$$

want unif. inf.

Lemma (L1): $g:\Omega\to[0,1]$, meas. we have

$$\mathbb{E}[|W_n(g) - W_m(g)|] \le \mathbb{E}[|W_n^\beta - W_m^\beta|].$$

HARRIS/FKG INEQUALITY

$M \ge 1$. $\vec{X} = (X_1, \dots, X_M)$ iid real-valued r.v.'s.

$x, y \in \mathbb{R}^M$, $x \ge y \Leftrightarrow \forall i \in [\![1,M]\!]\ x_i \ge y_i$.

$f:\mathbb{R}^M \to \mathbb{R}_+$ is increasing if $\forall x,y,\ x \ge y \Rightarrow f(x) \ge f(y)$

THM: If $f,g:\mathbb{R}^M \to \mathbb{R}_+$ are increasing ($\&$ in L^2), then $\mathbb{E} f(\vec{X}) g(\vec{X}) \ge \mathbb{E} f(\vec{X}) \cdot \mathbb{E} g(\vec{X})$

Proof (L1): (wlog $n \ge m$)

$$|x| = 2x_+ - x \qquad (x_+ = \max(x,0)).$$

STS: $\mathbb{E}[(W_n(g) - W_m(g))_+] \le \mathbb{E}[(W_n - W_m)_+]$.

$$(W_n - W_m)_+ \ge (W_n - W_m)_+ \mathbb{1}_{W_n(g) \ge W_m(g)}$$

$\boxed{\bar{g} := 1-g}$

$$= (W_n - W_m)\mathbb{1}_{W_n(g) \ge W_m(g)}$$

$$\overset{(=)}{\ge} (W_n(g) - W_m(g))_+ + (W_n(\bar g) - W_m(\bar g))\mathbb{1}_{W_n(g) \ge W_m(g)}.$$

$$\mathbb{E}[(W_n(\bar g) - W_m(\bar g))\mathbb{1}_{W_n(g) \ge W_m(g)} | \mathcal{F}_m] \ge 0. \quad (*)$$

CLAIM: $W_n(f)$ is an increasing function of $(\omega_{k,x})_{k \ge m+1, x\in\mathbb{Z}^d}$

$$W_n(f) = E[e^{(\cdots)} f(X)]$$

So $W_n(\bar g) - W_m(\bar g)$ is increasing so FKG $\Rightarrow (*)$

and $\mathbb{E}[W_m(\bar g) | \mathcal{F}_m] = W_m(\bar g) \Rightarrow$ Lemma

$\square$.

Want to estimate: (with $m = m_N = N^{1/4}$)

$$|E_N^{\beta,\omega}[\varphi(X^{(N)})] - Q(\varphi(B))|$$

$$\le |E_N^{\beta,\omega}[\varphi(X^{(N)})] - E_m^{\beta,\omega}[\varphi(X^{(N)})]| \ ①$$

$$+ |E_N^{\beta,\omega}[\varphi(X^{(N)})] - Q(\varphi(B))| \ ②$$

①: Now, $\mathbb{E}[|W_N(\varphi(X^{(N)})) - W_m(\varphi(X^{(N)}))|]$

by lemma $\le \mathbb{E}[|W_N - W_m|] \to 0$, when $N(\&m) \to \infty$.

As a consequence,

$$\left| \frac{W_N(\varphi(X^{(N)}))}{W_N} - \frac{W_m(\varphi(X^{(N)}))}{W_m} \right| \xrightarrow[\text{in } L^1]{N\to\infty} 0$$

②: consider X with law $P_m^{\beta\omega}$

Y SRW.

Will couple (X,Y) so that X,Y are indep. until time m, and

$(X_n - X_m)_{n\ge m} = (Y_n - Y_m)_{n \ge m}$

$\longrightarrow |X_n - Y_n| \le 2m \quad \forall n \ge 0$

$$|X^{(N)} - Y^{(N)}|_\infty \le \frac{2m}{\sqrt{N}} \le 2N^{-1/4}.$$

"Disorder only affects first few steps and so does not survive in the limit."

$\square$

$d \ge 3$, β small $\Longrightarrow W_N^\beta$ is bdd in L^2

$$E[(W_N^\beta)^2] = E[E^{\otimes 2}[\exp(\sum_{n=1}^{N}[\beta(\omega_{n,X_n^{(1)}} + \omega_{n,X_n^{(2)}}) - 2\lambda(\beta)])]]$$

(Fubini)

$$= E[\exp(\beta(\omega_{m,X_m^{(1)}} + \omega_{m,X_m^{(2)}}) - 2\lambda(\beta))]$$

$$= \begin{cases} e^{\lambda(2\beta) - 2\lambda(\beta)} =: \chi(\beta) & \text{if } X_m^{(1)} = X_m^{(2)} \\ 1 & \text{if } X_m^{(1)} \ne X_m^{(2)} \end{cases}$$

$$E[(W_N^\beta)^2] \overset{\le}{=} E^{\otimes 2}[(\chi(\beta))^{\sum_{n=1}^{\infty} \mathbb{1}_{X_m^{(1)} = X_m^{(2)}}}].$$

If $d \ge 3$, then $\sum_{n=1}^{\infty} \mathbb{1}(X_n^{(1)} = X_n^{(2)}) < \infty$ a.s.

Moreover, it is a geometric r.v. (by Markov).

$$\sup_N E[(W_N^\beta)^2] < \infty \text{ if } (\chi(\beta)-1) E^{\otimes 2}[\sum_{n=1}^{\infty} \mathbb{1}_{X_n^{(1)} = X_n^{(2)}}] < 1$$

VERY STRONG DISORDER

$$F(\beta) := \lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log W_N^\beta < 0.$$

Claim: F(β) exists

$\mathbb{E}[\log W_N^\beta]$ is super-additive, i.e.

$$\mathbb{E} \log W_{N+M}^\beta \ge \mathbb{E} \log W_N^\beta + \mathbb{E} \log W_M^\beta$$

Recall:

$$W_{N+M} = \sum_{x\in\mathbb{Z}^d} \hat{W}_N(x)\, \Theta_{N,x} W_M.$$

$$= W_N \sum_{x\in\mathbb{Z}^d} \mu_N(x)\, \Theta_{N,x}(W_M), \text{ where}$$

$$\mu_N(x) = P_N^{\beta\omega}(X_N = x) = \frac{\hat{W}_N(x)}{W_N}$$

$$\mathbb{E} \log W_{N+M}^\beta = \mathbb{E} \log W_N + \mathbb{E}[\log \sum_x \mu_N(x) \Theta_{N,x}(W_M)]$$

$$\ge \mathbb{E} \log W_N + \mathbb{E}[\sum_x \mu_N(x) \log \Theta_{N,x}(W_M)]$$

$$= \mathbb{E} \log W_N + \mathbb{E}[\mathbb{E}[\sum \mu_N(x) \log \Theta_{N,x} W_M | \mathcal{F}_N]]$$

$$= \mathbb{E} \log W_N + \mathbb{E}[\sum \mu_N(x) \mathbb{E} \log W_M] = \cdots \checkmark$$

Now, $\mathbb{E} \log W_N = 2 \mathbb{E} \log \sqrt{W_N} \le 2 \log \mathbb{E} \sqrt{W_N}$

$$\Longrightarrow F(\beta) \le \liminf_{n\to\infty} \frac{2}{N} \log \mathbb{E} \sqrt{W_N}$$

Recall $\sqrt{\sum a_i} \le \sum \sqrt{a_i}$.

$$\text{So } \sqrt{W_N} \le \frac{1}{(2d)^{N/2}} \sum_{x\in\Omega_N} \exp\left(\sum_{n=1}^{N} \frac{\beta}{2}\omega_{n,x_n} - \frac{\lambda(\beta)}{2}\right)$$

$$\mathbb{E}[\sqrt{W_N}] \le \frac{1}{(2d)^{N/2}} (2d)^N \exp\left(\lambda\left(\frac{\beta}{2}\right) - \frac{\lambda(\beta)}{2}\right)^N$$

$$= \left(\sqrt{2d} \exp\left(-\frac{\lambda(\beta)}{2} - \lambda\left(\frac{\beta}{2}\right)\right)\right)^N$$

if $\lambda(\beta) - 2\lambda(\beta/2) \to \infty$ when $\beta \to \infty$, then ✓.

(s.b. ∄ atom on ess sup wrt ω.)

Thm: If very strong disorder holds, then:

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} I_n > 0 \quad \Big| \quad I_n = \|\mu_n\|_2^2$$

Given $[\beta_1, \beta_2]$ there exists m, M such that

$$-mF(\beta) \le \frac{1}{N} \sum_{n=1}^{N} I_n \le -MF(\beta).$$

Thm: 2.25

$$\lim_{\beta \downarrow \beta_c} \frac{\log F(\beta)}{\log(\beta - \beta_c)} = \infty$$

So, $\forall k \ge 1$: $F(\beta) \le C_k(\beta - \beta_c)^k$ for $\beta \in [\beta_c, \beta_c + 1]$.

VERY STRONG DISORDER REGIME

$F(\beta) := \lim_{N\to\infty}\frac{1}{N}\log W_N^\beta$

VSD $\Longleftrightarrow$ $F(\beta) < 0$.

Thm (CH '02, CSY '03)

$\exists$ Fix $\beta_1 < \beta_2$ $\exists\, m, M \in (0,\infty)$ s.t. $\forall \beta\in[\beta_1,\beta_2]$

$$m|F(\beta)| \le \liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} I_n \le \limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} I_n \le M\,|F(\beta)|$$

$$I_n = \sum_{x\in\mathbb{Z}^d} P_{n-1}^{\beta,\omega}(X_n = x)^2$$

$\mu_n(x) = P_n^{\beta,\omega}(X_n = x)$, $P_{n-1}^{\beta,\omega}(X_n\in\cdot\cdot) = \underbrace{D}_{\text{generator of SRW}}\mu_{n-1}$

$$\frac{1}{2d}\|\mu_{n-1}\|_2^2 \le \|D\mu_{n-1}\|_2^2 \le \|\mu_{n-1}\|_2^2$$

Proof:
$$\log W_N = \sum_{n=1}^{N}\log\frac{W_n}{W_{n-1}}$$
$$A_N \leftarrow = \sum_{n=1}^{N}\log\frac{W_n}{W_{n-1}} - \mathbb{E}\left[\log\frac{W_n}{W_{n-1}}\Big|\mathcal{F}_{n-1}\right]$$
$$B_N \leftarrow + \sum_{n=1}^{N}\mathbb{E}\left[\log\frac{W_n}{W_{n-1}}\Big|\mathcal{F}_{n-1}\right]$$

Show: $\mathbb{E}\left[\log\frac{W_n}{W_{n-1}}\Big|\mathcal{F}_{n-1}\right] \asymp -I_n$.

Assume for simplicity that ω is bdd.

A_N: if ω bdd then $\left|\log\frac{W_n}{W_{n-1}}\right| \le K$ for some K.

By Azuma-Hoeffding,
$$\mathbb{P}\left(A_N \ge N^{3/4}\right) \le \exp\left(-cN^{1/2}\right)$$

and by Borel-Cantelli, $A_N/N \to 0$ a.s.

Recall $W_n = \sum_{x\in\mathbb{Z}^d}\hat{W}_{n-1}(x)\,\theta_{n-1,x}W_1$

$$\frac{W_n}{W_{n-1}} \circeq \sum_{x\in\mathbb{Z}^d}\mu_{n-1}(x)\,\underbrace{\theta_{n-1,x}W_1}_{\to (D\zeta_\beta)(n,x)}$$

Define $\zeta_\beta(n,x) = e^{\beta\omega_{n,x}-\lambda(\beta)}$

$$\overset{\text{"self-adjointness"}}{\circeq} \sum_{x\in\mathbb{Z}^d} D\mu_{n-1}(x)\,\zeta_\beta(n,x)$$

$\bar{\zeta}_\beta := \zeta_\beta - 1$

$\zeta_\beta \ge 1/K$.

$$\log\Big(1+\underbrace{\sum_{x\in\mathbb{Z}^d} D\mu_{n-1}(x)\,\bar{\zeta}_\beta(n,x)}_{\in[1/K-1,K]\ (\text{conv. comb. of } \bar{\zeta}_\beta)}\Big)$$

$$-Mu^2 \le \log(1+u)-u \le -mu^2,\quad u\in[1/K-1,K]$$

So
$$B_N = \mathbb{E}\Big[\log\Big(1+\sum_{x\in\mathbb{Z}^d}D\mu_{n-1}(x)\bar{\zeta}_\beta(n,x)\Big) - \underbrace{\sum_{x\in\mathbb{Z}^d}D\mu_{n-1}(x)\bar{\zeta}_\beta(n,x)}_{\mathbb{E}[\cdots]=0}\Big|\mathcal{F}_{n-1}\Big]$$
$$\asymp -\mathbb{E}\Big[\Big(\sum_{x\in\mathbb{Z}^d}D\mu_{n-1}(x)\bar{\zeta}_\beta(n,x)\Big)^2\Big|\mathcal{F}_{n-1}\Big]$$
$$= -\underbrace{\mathrm{Var}(\zeta_\beta(1,c))}_{\text{bdd from below}}\, I_n$$

Theorem JL25+

If $W_N \xrightarrow{N\to\infty} 0$ then $\lim_{N\to\infty}\frac{1}{N}\mathbb{E}[\log W_N] < 0$.

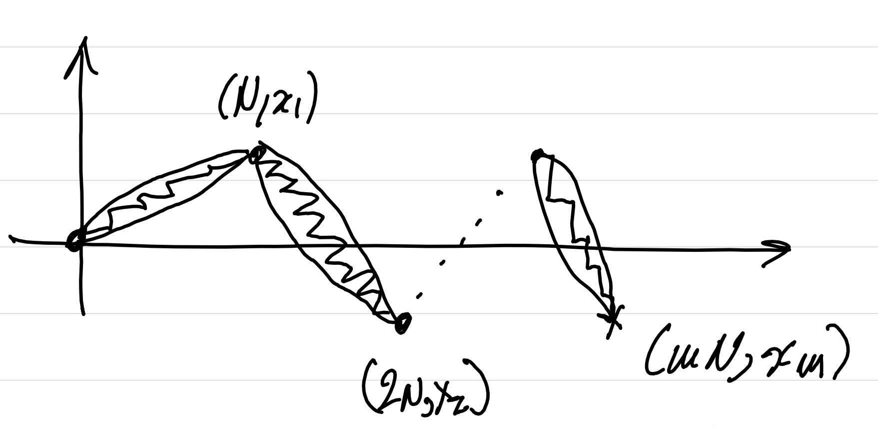
SB: $\frac{2}{N}\liminf_{N\to\infty}\log\mathbb{E}\left[\sqrt{W_N^\beta}\right] \ge F(\beta)$

Proposition: CV '07

If for some $N\ge 1$
$\mathbb{E}[\sqrt{W_N^\beta}] < (2N+1)^{-d} \Rightarrow F(\beta) < 0$ (*)

Proof: Fix N take $m\to\infty$ and WTS:
$\mathbb{E}[\sqrt{W_{Nm}}]$ decays exponentially in m if (*) holds.

$$W_{Nm} \circeq \sum_{x_1,\dots,x_m\in\mathbb{Z}^d}\mathbb{E}\Big[e^{\sum_{n=1}^{Nm}(\dots)}\mathbb{1}(X_N = x_1,\dots,X_{2N}=x_2,\dots,X_{Nm}=x_m)\Big]$$



$$\circeq \sum_{x_1,\dots,x_m\in\mathbb{Z}^d}\prod_{i=1}^{m}\theta_{(i-1)N,x_{i-1}}\underbrace{\hat{W}_N(x_i-x_{i-1})}$$

and taking square roots:
$$\sqrt{W_{Nm}} \le \sum_{x_1,\dots,x_m}\mathbb{E}\Big[\prod_{i=1}^{m}\sqrt{\quad}\Big]$$

(+ indep.)
$$\Longrightarrow \mathbb{E}[\sqrt{W_{Nm}}] \le \sum_{x_1,\dots,x_m\in\mathbb{Z}^d}\prod_{i=1}^{m}\mathbb{E}\Big[\sqrt{\hat{W}_N(x_i-x_{i-1})}\Big]$$
$$(y_i = x_i - x_{i-1}) = \Big(\sum_{\substack{y_1,\dots,y_m\in\mathbb{Z}^d\\(y)}}\underbrace{\mathbb{E}[\sqrt{\hat{W}_N(y)}]}\Big)^m$$
$$|y|\le N \qquad \lesssim (2N+1)^d\,\mathbb{E}[\sqrt{W_N}]$$

$d\tilde{\mathbb{P}}_N = W_N^\beta\, d\mathbb{P}$ (size-biased measure).

Lemma: $\mathbb{E}[\sqrt{W_N}] \le \sqrt{\mathbb{P}(A)} + \sqrt{\tilde{\mathbb{P}}_N(A^c)}$ for any A measurable.

Proof:
$$\mathbb{E}[\sqrt{W_N}] = \underbrace{\mathbb{E}[\sqrt{W_N}\mathbb{1}_A]}_{C\text{-}S} + \underbrace{\mathbb{E}[\sqrt{W_N}\mathbb{1}_{A^c}]}_{\text{Jensen}}$$
$$\le \dots$$

□

Find A_N which is unlikely under $\mathbb{P}$ and likely under $\tilde{\mathbb{P}}_N$

Lemma: if $W_\infty = 0$ then
$\mathbb{P}\left(\sup_{N\ge 1} W_N \ge u\right) \in \left[\frac{1}{Lu}, \frac{1}{u}\right]$ for some $L\ge 0$.

Proof: If ω is upper bdd $\frac{W_n}{W_{n-1}} \le L$

Let $\tau_u = \inf\{W_N > u\}$. By OST:
$$\mathbb{E}\Big[\overbrace{W_{N\wedge\tau_u}}^{\le Lu}\Big] = 1$$

Take $N\to\infty$ and by DCT
$$\mathbb{E}\left[W_{\tau_u}\mathbb{1}(\tau_u<\infty) + 0\cdot\mathbb{1}(\tau_u=\infty)\right] = 1.$$

Have $u\mathbb{P}(\tau_u<\infty) \le 1 \le Lu\cdot\mathbb{P}(\tau_u<\infty)$.

Lemma:
Given $\varepsilon>0$, $\exists\, C_\varepsilon > 0$ s.t. $\forall u>1$,
$\exists\, m_u \in [\![0, C_\varepsilon\log u]\!]$, $\mathbb{P}(W_{m_u}\ge u) \ge u^{1-\varepsilon}$

$\mathbb{P}[W_m \ge u] \le \frac{1}{u}$,

$$\tilde{\mathbb{P}}_m[W_m\ge u] = \mathbb{E}[W_m\mathbb{1}\{W_m\ge u\}] \overset{\text{"very likely under size-biased meas."}}{\ge} u^{-\varepsilon}$$

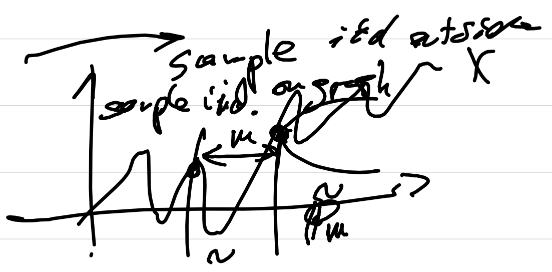
$u = N^{4d}$, m given by above Lemma for $\varepsilon = \frac{1}{12d}$ $(m\le C\log N)$

$\mathcal{A}_N := \{\exists (k,x)\in[\![0,N]\!]\times[\![-N,N]\!]^d,\ \theta_{k,x}W_m \ge N^{4d}\}$
"$\exists$ some point reachable from SRW st. have large partition function".

$$\text{Now, } \mathbb{P}(\mathcal{A}_N) \overset{\text{union bound}}{\le} (N+1)\cdot(2N+1)^d\,\mathbb{P}(W_m\ge N^{4d}) \le CN^{1-3d}.$$

$$\tilde{\mathbb{P}}_m[W_m \ge N^{4d}] \ge N^{-4d\varepsilon}$$

$$d\tilde{\mathbb{P}}_N = \frac{1}{(2d)^N}\sum_{x\in\Omega_N}\exp\Big(\sum_{n=1}^{N}\beta\omega_{n,x_n}-\lambda(\beta)\Big)\,d\mathbb{P}$$
$$=: \frac{1}{(2d)^N}\sum_{x\in\Omega_N} d\tilde{\mathbb{P}}_x$$



(here need $m \ll N$).

$\mathcal{A}_N^c = \{\forall (k,x): \theta_{k,x}W_m \le N^{4d}\}$

$$\subseteq \{\forall i\in[\![0,\tfrac{N}{m}-1]\!],\ \theta_{im,x_{im}}W_m \le N^{4d}\} \to B_N$$

$$\tilde{\mathbb{P}}_x(B_N) = \left(\tilde{\mathbb{P}}_m[W_m\le N^{4d}]\right)^{N/m} \le \underbrace{\left(1-N^{-1/3}\right)^{N/m}}_{\le\exp(-\sqrt{N})}.$$

□

Theorem: (COMETS, YOSHIDA '06).

$\exists \beta_c(d, \text{law } \omega)$ s.t.

$\beta < \beta_c$ WEAK DISORDER $\iff \lim_{n\to\infty} W_N^\beta = \begin{cases} W_\infty^\beta > 0 \\ -\text{''}- = 0 \end{cases}$

$\beta > \beta_c$ STRONG DISORDER

$$W_N^\beta = \overset{\swarrow \text{ SRW measure}}{E}\left[e^{\sum \beta\omega_{n,x_n} - \lambda(\beta)}\right].$$

Enough to show $\beta_1 < \beta_2$

$\text{strong dis.} \Rightarrow \text{strong dis.}$

Show $\beta \longmapsto \mathbb{E}(W_\infty^\beta)^\delta$ is decreasing, $\delta \in (0,1)$.

CLAIM: $\beta \longmapsto \mathbb{E}[\phi(W_N^\beta)]$ is decreasing, (assume ϕ' dec.)

$(\phi(u) = u^\delta)$

Indeed, $\frac{\partial}{\partial\beta}\mathbb{E}[\phi(W_N^\beta)]$ $(*)$

$$= \mathbb{E}\left[\frac{\partial}{\partial\beta}\phi(W_N^\beta)\right]$$

$$= \mathbb{E}[\phi'(W_N^\beta) \times Y_N^\beta].$$

$$Y_N^\beta := \frac{\partial}{\partial\beta} E\left[e^{\sum_{n=1}^{N} \beta\omega_{n,x_n} - \lambda(\beta)}\right]$$

$$= E\left[e^{\sum \beta\omega_{n,x_n} - \lambda(\beta)}\left(\sum \omega_{n,x_n} - \lambda'(\beta)\right)\right]$$

So, $(*) = E\left[\mathbb{E}\left[\phi'(W_N^\beta)\cdot\left(\sum \omega_{n,x_n} - \lambda'(\beta)\right)e^{\sum\beta\omega_{n,x_n}-\lambda(\beta)}\right]\right]$

$$\mathbb{E}\left[e^{\sum\beta\omega_{n,x_n} - \lambda(\beta)}\right] = 1.$$

$\phi'(W_N^\beta)$ is decreasing wrt environment.

$\sum \omega_{n,x_n} - \lambda'(\beta)$ increasing.

So $\mathbb{E}\left[\phi'(W_N^\beta)\cdot\left(\sum\omega_{n,x_n} - \lambda'(\beta)\right)e^{\sum\beta\omega_{n,x_n}-\lambda(\beta)}\right]$

$\leq \mathbb{E}\left[\phi'(W_N^\beta)\cdot e^{\sum\beta\omega_{n,x_n} - \lambda'(\beta)}\right]\cdot\mathbb{E}\left[\left(\sum\omega_{n,x_n} - \lambda'(\beta)\right)e^{\cdots}\right]$

$= 0$

since $\mathbb{E}\left[(\omega_{i,x_i} - \lambda'(\beta))\cdot e^{\beta\omega_{n,x_n} - \lambda(\beta)}\right]$

$\lambda(\beta) = \log \mathbb{E}[e^{\beta\omega}]$

$$\lambda'(\beta) = \frac{\mathbb{E}[\omega e^{\beta\omega}]}{\mathbb{E}[e^{\beta\omega}]}.$$

We still NTS that

$\sup_{\beta\in[\beta_1,\beta_2]} \frac{\partial}{\partial\beta}\phi(W_N^\beta)$ is integrable

Let's assume that $\phi'(u) \leq u + u^{-1}$

So $\left|\frac{\partial}{\partial\beta}\Phi(W_N^\beta)\right| = \Phi'(W_N^\beta)\cdot Y_N^\beta \leq W_N^\beta\cdot|Y_N^\beta| + (W_N^\beta)^{-1}\cdot|Y_N^\beta|$

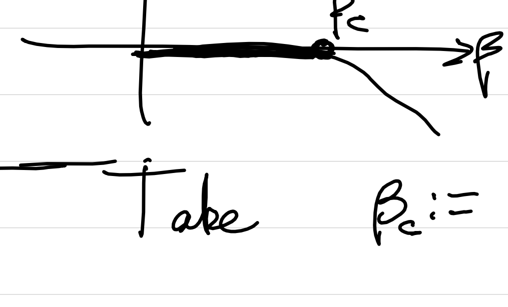
$$(W_N^\beta)^{-1} \leq \mathbb{E}\, e^{-\sum\beta\omega_{n,x_n} + \lambda(\beta)} \leq e^{n\lambda(\beta)}\, E e^{\beta\sum|\omega_{n,x_n}|}$$

$$\text{So} \left(\sup_{\beta\in[\beta_1,\beta_2]} (W_N^\beta)^{-1}\right)^2 \leq \left(\sup_{\beta\in[\beta_1,\beta_2]} e^{n\lambda(\beta)}\right)^2\cdot E e^{2\beta_2\sum|\omega_{n,x_n}|}$$

$$\Longrightarrow \sup_{\beta\in[\beta_1,\beta_2]} (W_N^\beta)^{-1} \in L^2(\mathbb{P}).$$

If $\phi(u) = \log u \Longrightarrow \beta \mapsto \frac{\mathbb{E}\log W_N^\beta}{N}$ is non-inc.

Send $N \longrightarrow \infty \Longrightarrow F(\beta)$ is non-increasing.



Take $\beta_c := \inf\{\beta \in \mathbb{R}_+ : \mathbb{E}W_\infty^\beta = 0\}$

FKG/Harris Ineq:

$X_1, X_2, \ldots, X_k$ r.v.'s independent

$f, g : \mathbb{R}^k \longrightarrow \mathbb{R}$ increasing.

$$\Longrightarrow \mathbb{E}[f(\underbrace{X_1, \ldots, X_k}_{X})\cdot g(\underbrace{X_1, \ldots, X_k}_{X})] \quad (\dagger)$$

$$\geq \mathbb{E} f(X) \cdot E\, g(X).$$

Proof: by induction.

$\underline{k = 1:}$ WTS $\mathbb{E} f(X_1) g(X_1)] \geq \mathbb{E} f(X_1)\cdot \mathbb{E} g(X_1)$.

let Y_1 be an indep. copy of X_1.

Consider $\mathbb{E}[(f(X_1) - f(Y_1))(g(X_1) - g(Y_1))]$

WTS $= 2\mathbb{E}[f(X_1) g(X_1)] - 2\mathbb{E}[f(X_1)]\cdot\mathbb{E}[g(X_1)]$

≥ 0

which is true by monotonicity (of f, g).

Now ind. step: condition on X_1:

$$(\dagger) = \overset{①}{\overbrace{Cov(\mathbb{E}[f(X)|X_1], \mathbb{E}[g(X)|X_1])}} + \mathbb{E}[\underbrace{Cov(f(X), g(X)|X_1)}_{②}]$$

So ① ≥ 0 by base case & conditioning.

② ≥ 0 by ind. hyp. & conditioning.

$\square$