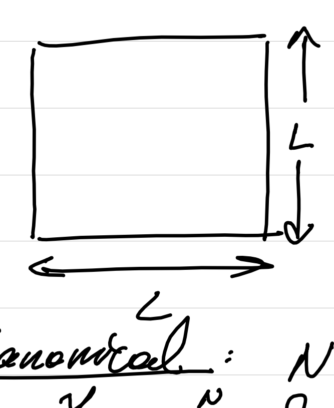


$Q_L = (0,L)^d$
$L \gg 1$
x_i iid points uniformly in Q_L

· Canonical: $N = L^d$ points
$\gamma = \sum_{i=1}^{N} \delta_{X_i}$

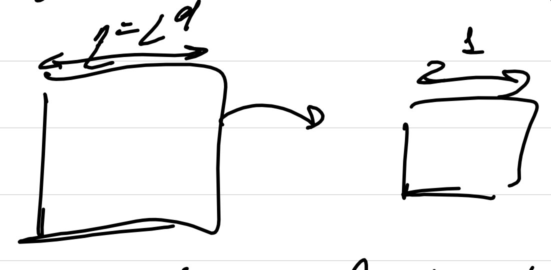
Q: $\mathbb{E}[\frac{1}{L^d} W^p(\gamma, 1)]$? (1 = leb. meas.) Opt. Trans. Plan?

Caracciollo - Lucibello - Parisi - Sicuro.
variant: bi-partite, Y_j iid...
$\mathbb{E}[\frac{1}{L^d} W^p(\gamma, \lambda)]$, $\lambda = \sum \delta_{Y_j}$

- Grand canonical: $N \sim$ Poisson parameter L^d, γ Poisson Point Process. (PPP)
$\Omega \subset Q_L$ $\gamma \llcorner \Omega$ PPP parameter $L^d \cdot |\Omega|$.

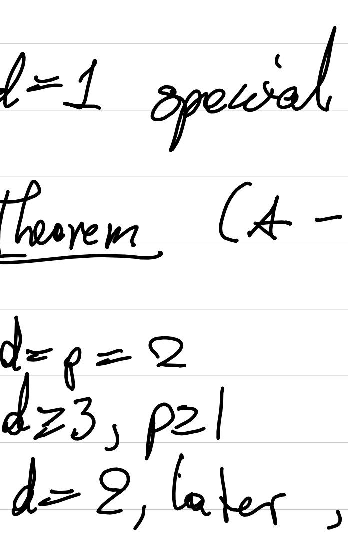
· If $L^d \gg 1$, N has very good concentration
go from determ. N to random "easily"

Scaling:



$X_i = Y L_i \to Y_i$ unif. distributed in Q_1.
$\mathbb{E}[\frac{1}{L^d} W^p(\gamma, 1)] = n^{p/d - 1} \mathbb{E}[W^p(\gamma_n, n)]$
$\gamma_n = \sum \delta_{Y_i}$

Heuristics: distance between $X_i \sim 1$
$\mathbb{E}[\frac{1}{L^d} W^p(\gamma, 1)] \sim 1$?



$\Rightarrow \mathbb{E}[\ ] \gtrsim 1$

Theorem (AKT, ...)
$\mathbb{E}[\frac{1}{L^d} W^p(\gamma, 1)] \sim \begin{cases} 1, & \text{if } d \geq 3 \\ (\log L)^{p/2}, & d = 2 \end{cases}$

$d=1$ special $\Rightarrow$ not discussed!

Theorem (A-S-T, B-B, D-S-S, G-T)

$d = p = 2$ $\quad \mathbb{E}[\frac{1}{L^2 \log L} W^2(\cdot)] \to \frac{1}{2\pi}$
$d \geq 3$, $p \geq 1$ $\quad \mathbb{E}[\frac{1}{L^d} W^p(\cdot)] \to b^{\infty}_{p,d} \in (0,\infty)$.
$d = 2$, later, now $d \geq 3$

Notation & Preliminaries:

γ, λ measures on $\mathbb{R}^d$ $\gamma(\mathbb{R}^d) = \lambda(\mathbb{R}^d)$
$W^p(\gamma, \lambda) = \min_{\pi} \{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^p d\pi(x,y), \pi_1 = \gamma, \pi_2 = \lambda \}$.

$W^p_\Omega(\gamma, \lambda) = W^p(\gamma \llcorner \Omega, \lambda \llcorner \Omega)$.

$W^p_\Omega(\gamma, k) = W^p(\gamma, \frac{\gamma(\Omega)}{|\Omega|} dx)$.
$k = \frac{\gamma(\Omega)}{|\Omega|}$.

Main properties:

· triangle inequality, $W(\gamma, \lambda) \leq W(\gamma, \gamma') + W(\gamma', \lambda)$
· Young $\Rightarrow W^p(\gamma, \lambda) \leq (1+\varepsilon) W^p(\gamma, \gamma') + \frac{C}{\varepsilon^{p-1}} W^p(\gamma', \lambda)$.
• subadditivity: $W^p(\gamma_1 + \gamma_2, \lambda_1 + \lambda_2)$
$\leq W^p(\gamma_1, \lambda_1) + W^p(\gamma_2, \lambda_2)$.

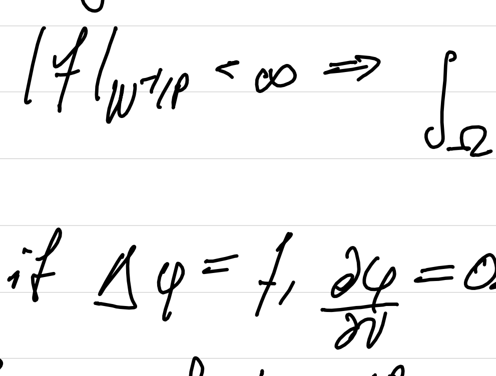
⚠ $\gamma_1(\mathbb{R}^d) = \lambda_1(\mathbb{R}^d)$
$\gamma_2(\mathbb{R}^d) = \lambda_2(\mathbb{R}^d)$.

Benamou - Brenier formula

Ω bounded Lipschitz ($\Omega = \mathbb{T}_L = (\mathbb{R}/L\mathbb{Z})^d$)

$W^p_\Omega(\gamma, \lambda) \leq \inf_{(\rho, j)} \{ \int_0^1 \int_\Omega \frac{1}{\rho^{p-1}} |j|^p : \partial_t \rho + \operatorname{div} j = 0$ in Ω, $j \cdot \nu = 0$ on $\partial\Omega$, $\rho_0 = \gamma, \rho_1 = \lambda \}$

• $j = v\rho \Rightarrow \int_0^1 \int_\Omega |v|^p d\rho dt$
• Cont. eq $\forall \xi \in C^\infty(\bar{\Omega} \times [0,1])$,
$\int_0^1 \int_\Omega \rho \partial_t \xi + j \cdot \nabla \xi = \int \xi_1 d\lambda - \int \xi_0 d\gamma$
· Ω convex or $\Omega = \mathbb{T}_L \Rightarrow$ equality



Sobolev spaces:

$|f|_{W^{-1,p}} = \sup_{|\nabla \varphi|_{L^{p'}} \leq 1} \int_\Omega \varphi f$
By duality: $|f|^p_{W^{-1,p}} = \min \{ \int_\Omega |b|^p, \operatorname{div} b = f$ in Ω, $b \cdot \nu = 0$ on $\partial\Omega$
Rk: $|f|_{W^{-1,p}} < \infty \Rightarrow \int_\Omega f = 0 \Rightarrow$ can test with $\int \varphi = 0$.

Rk: if $\Delta \varphi = f$, $\frac{\partial \varphi}{\partial \nu} = 0 \Rightarrow b = \nabla \varphi$ as competitor
$|f|^p_{W^{-1,p}} \leq \int_\Omega |\nabla \varphi|^p \rightsquigarrow$ PDE to get estimates.

Prop: Ω convex bounded $\int_\Omega f = 0$.
$|f|_{W^{-1,p}} \lesssim \operatorname{diam}(\Omega) \cdot |f|_{L^p}$

Proof: by scaling $\operatorname{diam}(\Omega) = 1$, $\int \varphi = 0$.
$\int \varphi f \leq (\int |f|^p)^{1/p} (\int |\varphi|^{p'})^{1/p'}$
$\underset{\text{Poincaré}}{\lesssim} (\int |f|^p)^{1/p} \cdot (\int |\nabla \varphi|^{p'})^{1/p'}$
for convex sets Poincaré constant depends only on $\operatorname{diam}(\Omega) \Rightarrow \sup_{|\nabla \varphi|_{L^{p'}} \leq 1} \int \varphi f \lesssim (\int |f|^p)^{1/p} = \|f\|_{L^p}$.
□

Prop: Ω bounded, Lip boundary, $\lambda \ll$ Leb, $\inf(\lambda) > 0$
$W^p_\Omega(\gamma, \lambda) \lesssim \frac{1}{(\inf \lambda)^{p-1}} |\gamma - \lambda|^p_{W^{-1,p}}$

Remark: if $p = 2$, Peyré

Proof: $W(\gamma, \lambda) \leq W(\gamma, \frac{\gamma+\lambda}{2}) + W(\frac{\gamma+\lambda}{2}, \lambda)$
$\leq W(\frac{\gamma}{2}, \frac{\lambda}{2}) + W(\frac{\gamma+\lambda}{2}, \lambda)$
$= \frac{1}{2^{1/p}} W(\gamma, \lambda) + W(\frac{\gamma+\lambda}{2}, \lambda)$

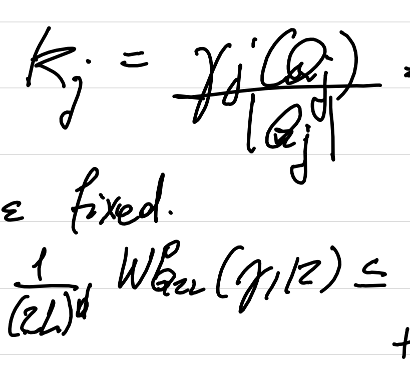
$\Rightarrow W^p(\gamma, \lambda) \lesssim W^p(\frac{\gamma+\lambda}{2}, \lambda)$

BB $\Rightarrow \forall b$ s.t. $\operatorname{div} b = \frac{\gamma+\lambda}{2} - \lambda = \frac{1}{2}(\gamma - \lambda)$
$b \cdot \nu = 0$ on $\partial\Omega$.
$j = b$, $\rho = t\lambda + (1-t)(\frac{\gamma+\lambda}{2})$.
$\partial_t \rho = \lambda - (\frac{\gamma+\lambda}{2}) = \operatorname{div} b$

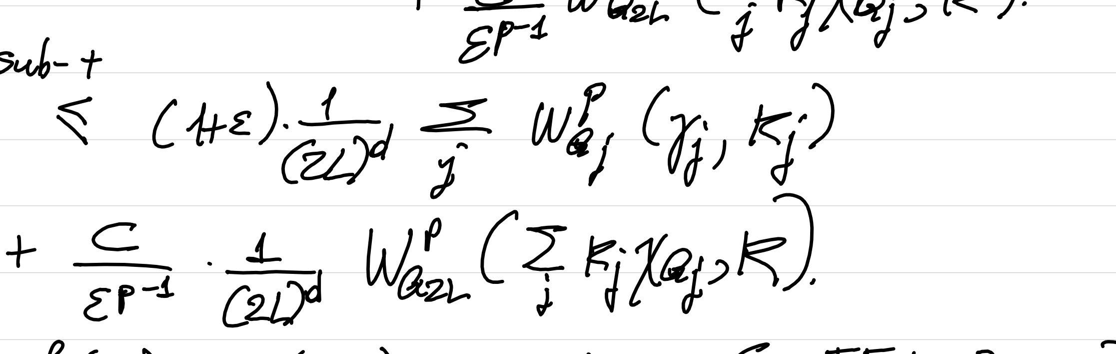
$\inf \rho_t \geq \frac{\inf \lambda}{2} \Rightarrow W^p(\gamma, \lambda) \lesssim \frac{1}{(\inf \lambda)^{p-1}} \int_\Omega |b|^p$
$\Rightarrow \min$ in $b \Rightarrow W^p(\gamma, \lambda) \lesssim \frac{1}{(\inf \lambda)^{p-1}} \cdot |\gamma - \lambda|^p_{W^{-1,p}}$

Corollary: Ω convex, bounded $\inf \lambda > 0$
$W^p_\Omega(\gamma, \lambda) \lesssim \frac{\operatorname{diam}(\Omega)^p}{(\inf \lambda)^{p-1}} \int_\Omega |\gamma - \lambda|^p$

Proof: that $b_{p,d}(L) = \mathbb{E}[\frac{1}{L^d} W^p_{Q_L}(\gamma, k)] \xrightarrow[L \to \infty]{} b^\infty_{p,d}$
p, d fixed have, γ is PPP, $k = \frac{\gamma(Q_L)}{|Q_L|}$



$\gamma_j = \gamma \llcorner Q_j$, PPP parameter L^d.



$k_j = \frac{\gamma_j(Q_j)}{|Q_j|} = \frac{\gamma(Q_j)}{|Q_j|}$

ε fixed.
$\frac{1}{(2L)^d} W^p_{Q_{2L}}(\gamma, k) \leq (1+\varepsilon) W^p_{Q_{2L}}(\gamma, \sum_j k_j \chi_{Q_j})$
$+ \frac{C}{\varepsilon^{p-1}} W^p_{Q_{2L}}(\sum_j k_j \chi_{Q_j}, k)$.
sub-+
$\leq (1+\varepsilon) \cdot \frac{1}{(2L)^d} \sum_j W^p_{Q_j}(\gamma_j, k_j)$
$+ \frac{C}{\varepsilon^{p-1}} \cdot \frac{1}{(2L)^d} W^p_{Q_{2L}}(\sum_j k_j \chi_{Q_j}, k)$.
$f(2L) \leq (1+\varepsilon) \frac{1}{2^d} \sum_j f(L) + \frac{C}{\varepsilon^{p-1}} \mathbb{E}[\frac{1}{L^d} W^p(\quad)]$
$= (1+\varepsilon) \cdot f(L) + \frac{C}{\varepsilon^{p-1}} \mathbb{E}[\quad]$.

Global term: $\frac{1}{L^d} W^p(\sum_j k_j \chi_{Q_j}, k) \lesssim \frac{1}{L^d}$
$L^p \sum_j |Q_j| \cdot |k_j - k|^p$
$\lesssim L^p \cdot |k_1 - k|^p$
$\lesssim L^p \cdot (|k_1 - 1|^p + |k - 1|^p)$.
$k_1 - 1 = \frac{\gamma(Q_1)}{L^d} - 1 = \frac{\gamma(Q_1) - L^d}{L^d} \sim L^{-d/2}$
$\mathbb{E}[|k_1 - 1|^p] \lesssim L^{-pd/2} \Rightarrow \mathbb{E}[\frac{1}{L^d} W^p(\cdot)] \leq L^{p(1-d/2)}$

$f(2L) \leq (1+\varepsilon) f(L) + \frac{C}{\varepsilon^{p-1}} \frac{1}{L^{p/2(d-2)}}$

$\Rightarrow f(L) \lesssim 1 \Rightarrow$ optimise
$f(2L) \leq f(L) + \frac{C}{L^{\frac{d}{2}-1}}$

Prop: if $\forall n \in \mathbb{N}$ $\forall t > 0$
$f(nt) \leq f(t) + Ct^{-\alpha} \Rightarrow \lim_{t \to \infty} f(t)$ exists.

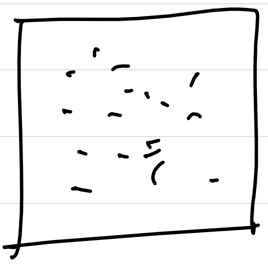
$\Rightarrow$ extends TSP_p, $p < d$.
(Travelling Salesman Problem).

Case of densities
Ω Lipschitz connected $\rho \in C^\alpha(\Omega)$, $\alpha \in [d/2, 2]$. X_i iid $\sim \rho$, $\gamma = \sum_{i=1}^{n} \delta_{X_i}$

$\limsup_{n \to \infty} \frac{1}{n^{1-p/d}} \mathbb{E}[W^p_\Omega(\gamma, n\rho)] \leq b^\infty_{p,d} \int_\Omega \rho^{1-p/d}$

LECTURE 2

$\boxed{p=2}$



$\gamma^L = \sum_{i=1}^{L^d} \delta_{X_i}$

$W_{Q_L}^2(\gamma, 1) = \int |T-x|^2$

Q_L

Ansatz of Caracciolo and al.

$T\#1 = \gamma$, $T = \nabla\psi$ convex ψ.

$\det \nabla^2\psi = \gamma$, $L >> 1$. $T \sim Id$, $\psi = \frac{1}{2}|x|^2 + \varphi$

$\det(Id + \nabla^2\varphi) = 1 + \Delta\varphi = \gamma$.

$\boxed{\Delta\varphi = \gamma - 1} \longrightarrow \int |T-x|^2 = \int |\nabla\varphi|^2$

$(= +\infty$ for $\gamma = \sum \delta_{x_i})$.

Based on this many predictions:

$d=2$ $\quad \mathbb{E}[\frac{1}{L^2} W_{Q_L}^2(\gamma, L^d)] - \frac{\log L}{4\pi} \xrightarrow{?} \xi \in \mathbb{R}$.

Remark: $W^2(\gamma,\lambda) \lesssim \|\gamma - \lambda\|^2_{W^{-1,2}} \le \int |\nabla\varphi|^2$

$(\Delta\varphi = \gamma - \lambda)$

$\mathbb{T}_L = (\mathbb{R}/L\mathbb{Z})^2$, η radially symmetric conv. kernel $\eta \ge 0$ $\int\eta = 1$, spt $\eta \subset B_1$. $\forall r > 0$

$\eta_r = \frac{1}{r^2}\eta(\cdot/r)$

$\xi_r(x) = \int_{\mathbb{T}_L} \eta_r(y-x)\xi(y)\,dy$

φ, $\int\varphi = 0$ solution of $-\Delta\varphi = \gamma - 1$ on $\mathbb{T}_L$ $\quad (-\Delta\varphi_r = \gamma_r - 1)$.

$\beta(r) = \begin{cases} 1+\log r, & d=2 \\ 1, & d\ge 3\end{cases}$

Theorem (G-H), $\exists r_*$ with

$$\mathbb{E}\left[\exp\left(c \frac{r_*^2}{\beta(r_*)}\right)\right] < +\infty$$

π Opt. Transp. plan for $W^2_{\mathbb{T}_L}(\gamma,1)$. Thus, $r_* < r < L$

(1) $\left|\int_{\mathbb{T}_L\times\mathbb{T}_L} \eta_r(x)(x-y-\nabla\varphi_r(0))\, d\pi\right| \lesssim r_*^2 \cdot \frac{\beta(r/r_*)}{r}$

and (2) $\sup\{|x-y-\nabla\varphi_r(0)|,\ (x,y)\in \operatorname{spt}\pi \cap (B_r\times\mathbb{R}^d)\}$

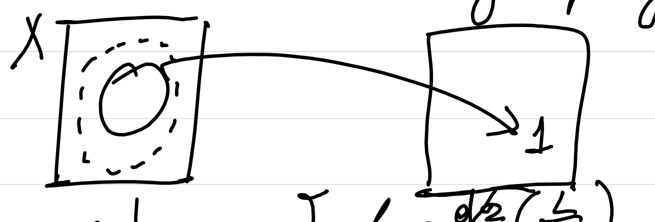
$\lesssim r\cdot\left(\frac{r_*^2\,\beta(r/r_*)}{r^2}\right)^{\frac{1}{d+2}}$

Remarks: (1) $\Rightarrow$ $x-y = \pi(y) - y \sim \nabla\varphi_r(0)$

weak (topology) sense $\sim |(T-x)_r(0) - \nabla\varphi_r(0)|$

$\lesssim 1/r$.

(2) closeness in strong topology



$|\nabla\varphi_r(0)| \sim \begin{cases} \log^{d/2}(\frac{L}{r}), & d=2\\ \frac{1}{r^{d/2(d-2)}}, & d\ge 3.\end{cases}$

(1)