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M2R Project report: Darboux Transformation
Group 3

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Abstract

This is a report of our group's study of Darboux Transformaions and its role in the spectral analysis of one dimensional Schrödinger operators. This method allows us to find the eigenvalues and eigenfunctions of such an operator via iteration in an algebraic manner.

We begin with some preliminary theory about the Hilbert spaces and linear operators of functions thereof. After developing this formalism, we will discuss successful applications of this approach for three Schrödinger equations (Schrödinger operator eigenvalue problems) with the method of Darboux Transformaions, which are as follows: the simple harmonic oscillator equation, the equation with reflectionless potential equation, and the equation with Coulomb potential. These problems have far reaching implications in the field of Quantum Mechancis and mathematical physics more broadly and are thus of fundamental physical interest.

Contents

1 Background

1.1 Hilbert Spaces

In this report, Hilbert spaces play a central role and are crucial in any analysis of the Schrodinger equation. Thus, we present a definition and some basic properties and constructions that will prove useful.

A Hilbert space is a vector space H with an inner product $\langle .,. \rangle : H \times H \to \mathbb{C}$ that becomes a complete normed vector space when equipped with the norm $||f|||_H = \sqrt{\langle f, f \rangle}$ [17]. We introduce some notions that are going to be relevant later on.

On a Hilbert space H, there are two types of convergence of a sequence of vectors $(f_n)_{n\in\mathbb{N}}$ that are commonly found in the literature [10]. First, we say that the sequence above converges in the **weak** sense to a vector $f \in H$ if and only if for any $u \in H$,

$$\langle u, f_n \rangle \to \langle u, f \rangle$$

as $n \to \infty$ in the usual sense, and this will be denoted as $f = w - \lim_{n\to\infty} f_n$. This is well defined since it its uniqueness is a direct consequence of the inner product on H. Similarly, if the sequence above is such that

$$||f - f_n||_H \to 0$$

as $n \to \infty$, we say that f is the **strong** limit of the sequence above denoted by$f = s - \lim_{n\to\infty} f_n$. Uniqueness follows from the standard properties of the norm, hence this limit is well-defined. It is also easy to show that strong convergence implies weak convergence.

In our analysis we will almost exclusively consider the space $L^2(\mathbb{R})$ that is defined below.

1.2 L^2 spaces

The L^2 space is an inner product space of measurable functions, with the inner product of two measurable functions f, g is defined as

$$(f, g) = \int_{-\infty}^{\infty} f(x)g^*(x)dx$$

[18]
and the norm of a function f is defined as $(\|f\|)^2 = \int_{-\infty}^{\infty} \mid f(x) \mid^2 dx$.
It it not difficult to verify that the above definitions are indeed inner product and norm.
The inner product under this definition is left-linear and symmetric under conjugation by the definition of integrals; and $(f, f) = \int_{-\infty}^{\infty} f(x)^2 dx \geq 0$ by the

property of integrals.
And for the norm, the non-negativity follows from the properties of integral, and $\|\lambda f\| = \sqrt{\int_{-\infty}^{\infty} \mid \lambda f(x) \mid^2 dx} = \lambda \sqrt{\int_{-\infty}^{\infty} \mid f(x) \mid^2 dx} = \lambda \|f\|$, and the triangle inequality can be proved by Minkowski's inequality for L^2.[14]

It is a standard result that the space L^2 is a Hilbert space, where two functions being identified if they disagree on a set of measure zero. Please refer to literature [14] for a proof of the completeness of L^p space, where p is a positive number.
It is also noteworthy to mention that the $L^2(\mathbb{R})$ space has a subset, namely that of smooth functions with compact support denoted by $C_c^\infty(\mathbb{R})$ that is dense, meaning that for all $u \in L^2(\mathbb{R})$, there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ such that

$$||f_n - u|| \to 0$$

as $n \to \infty$. Proofs of this standard result can be found in the literature [15].

1.3 Self-Adjoint linear operators

Suppose we consider a self-adjoint linear operator, that is a linear map

$$T : \mathcal{D}(H) \to H$$

in the usual sense, where $\mathcal{D}(H)$ can be any dense (in the topology induced by the norm $||v||_H = \langle v, v\rangle$) subset of the Hilbert space H under consideration, endowed with the following property:

$$\langle Tx, y\rangle = \langle x, Ty\rangle$$

for all x, y in H. That is T is its own adjoint, T^* which satisfies

$$\langle Tx, y\rangle = \langle x, T^* y\rangle$$

for all x, y in H. Existence and uniqueness of the adjoint for bounded linear operators T, bounded in the sense that $||Tx|| \leq M||x||$ for some $M < \infty$ for all x in H, is guaranteed by the Riesz Representation Theorem, that can be found in any reference on the topic [17].

We proceed with an important definition: the **spectrum** $\sigma(T)$ of a self-adjoint linear operator is the set given by the following characterisation due to Weyl [19]: $\lambda \in \mathbb{R}$ is in $\sigma(T)$ if and only if there exists a sequence $(\psi_n)_{n\in\mathbb{N}} \subseteq H$ with normalisation condition $||\psi_n|| = 1, \forall n \in \mathbb{N}$ such that

$$\|T\psi_n - \lambda\psi_n|| \to 0$$

as $n \to \infty$.

The **essential** spectrum of $T, \sigma_{ess}(T)$ is exactly as above, except for the fact that the sequence is a singular one, that is it has no convergent subsequence. It immediately follows that $\sigma_{ess}(T) \subseteq \sigma(T)$.

Furthermore, the **discrete** spectrum of $T, \sigma_{disc}(T)$ is defined as the compliment of the essential spectrum in the regular spectrum, that is $\sigma(T) \backslash \sigma_{ess}(T)$. Intuitively, it corresponds to the set of normal eigenvalues of a linear operator in the finite dimensional setting [17]. This intuition turns out to be true but this will not be explored here.

Next we will discuss an important result from [5] regarding the stability of the essential spectrum $\sigma_{ess}(T)$ under a 'compact perturbation' by an operator B. Let us be more precise and define the above kind of perturbation: consider a **bounded, self-adjoint and compact** linear operator B, where by compact we mean that for all bounded subsets W (bounded by the norm) of the Hilbert space H, one has that the closure of $B(W)$ is compact in the topology induced by the inner-product norm. Then, $B+T$ is self-adjoint and

$$\sigma_{ess}(B+T) = \sigma_{ess}(T)$$

By linearity of T and B, it is plain that $B+T$ is self-adjoint. Suppose now that $\lambda \in \sigma_{ess}(T)$, by definiton, there exists a singular sequence $(\psi_n)_{n\in\mathbb{N}}$ that converges weakly to 0. Since B is compact, it follows that $s - \lim\limits_{n\to\infty} B\psi_n = 0$. Thus, by the algebra of limits, it follows that $s - \lim\limits_{n\to\infty} (T+B)\psi_n - \lambda\psi_n = 0$ yielding $\lambda \in \sigma_{ess}(T+B)$ and $\sigma_{ess}(T) \subseteq \sigma_{ess}(T+B)$. The reverse inclusion is obtained by applying the above argument to $T+B$ and $-B$ and checking the conditions apply.

1.4 Schrödinger Operators

In this report, we will be considering operators of the form

$$-\frac{d^2}{dx^2} + V(x) : \mathcal{D}(H) \to H$$

for some $\mathcal{D}(H)$ dense in $H = L^2(\mathbb{R})$, typically, we will consider $C_c^\infty(\mathbb{R})$. Consider the expression $\langle \frac{d}{dx} f, g\rangle$. Integration by parts and compact supportedness of f and g yields that

$$\left\langle \frac{d}{dx} f, g \right\rangle = \left\langle f, -\frac{d}{dx} g \right\rangle$$

Thus, $\left(\frac{d}{dx}\right)^* = -\frac{d}{dx}$ and by repeated application of the above $\left(\frac{d^2}{dx^2}\right)^* = \frac{d^2}{dx^2}$. Since, we will assume that $V(x)$ is real-valued, by linearity, it follows that H is self-adjoint.

Perhaps rather surprisingly, according to [13], there are a few properties about the spectrum of the above operators that can be deduced from the limit of $V(x)$ as $|x| \to \infty$. The first case is when

$$V(x) \to \infty$$

as $|x| \to \infty$, where the spectrum of the operator $T = -\frac{d^2}{dx^2} + V(x)$

$$\sigma(T) = \sigma_{disc}(T)$$

Another important case that will crop up in later analysis is when

$$V(x) \to 0$$

$|x| \to \infty$ for $V(x)$ continuous, where one has

$$\sigma_{ess}(T) = [0, \infty)$$

2 Method

The method we mainly use is Darboux transformation, considering the one-dimensional Schrödinger equation (Sturm-Liouville equation):

$$-\psi_{xx} + u(x)\psi = \lambda\psi$$

If $\psi(x, \lambda)$, $\phi(x, \lambda) \in C^2$ are solutions of the above equation, then the Darboux transformation (DT) $\psi \to \psi[1]$ of the arbitrary solution is defined [9] by

$$\psi[1] = \psi_x - \sigma_1\psi; \quad \sigma_1 = \frac{\phi_{1x}}{\phi_1}$$

Darboux's theorem states that the function $\psi[1]$ is the solution of the differential equation

$$-\psi_{xx}[1] + u[1]\psi[1] = \lambda\psi[1],$$

where

$$u[1] = u - 2\sigma_{1x}$$

[9]
From Darboux's theorem, the one-dimensional Schrödinger equation (Sturm-Liouville equation) $-\psi_{xx}+u(x)\psi = \lambda\psi$ is covariant with respect to the Darboux transformation

$$\psi \to \psi[1], \quad u \to u[1]$$

Being covariant, the Darboux transformation can be iterated an arbitrary number of times to produce new solvable equations. The iterated Darboux transformation is expressed in Wronskian determinant W :

$$W(f_1, f_2, ..., f_k) = detA, \quad A_{ij} = \frac{d^{i-1}f_i}{dx^{i-1}}, \quad i, j = 1, 2, ..., k.$$

[9]
The Crum theorem states that the function

$$\psi[N] = \frac{W(\psi_1, \psi_2, ..., \psi_N, \psi)}{W(\psi_1, \psi_2, ..., \psi_N)},$$

is the solution of the differential equation

$$-\psi_{xx}[N] + u[N]\psi[N] = \lambda\psi[N],$$

with potential $u[N] = u - 2\frac{d^2}{dx^2} lnW(\psi_1, \psi_2, ..., \psi_N)$. In case $N = 1$, the Darboux theorem follows.[9]

3 Properties of Creation Annihilation Operator

Let us consider Schrödinger's equation $-\psi'' + u\psi = \lambda\psi$ for which we have a known eigenvalue λ_1 with its corresponding eigenfunction ψ_1. Hence it satisfies the following equation

$$-\psi_1'' + u\psi_1 = \lambda_1\psi_1.$$

We consider the following two operators:

- creation operator: $Q = \frac{d}{dx} - \frac{\psi_1'}{\psi_1}$
- annihilation operator: $Q^* = -\frac{d}{dx} - \frac{\psi_1'}{\psi_1}$.

We have constructed them such that Q and Q^* are adjoint operators, satisfying $(Q\psi, \phi) = (\psi, Q^*\phi)$, for all the functions $\psi, \phi \in L^2(\mathbb{R})$ and the inner product defined in 1. We compute the value of QQ^* and Q^*Q:

$$\begin{aligned}
QQ^*\psi &= \left(\frac{d}{dx} - \frac{\psi_1'}{\psi_1}\right)\left(\frac{d}{dx} - \frac{\psi_1'}{\psi_1}\right)\psi = -\psi'' + \frac{d}{dx}\left(\frac{\psi_1'}{\psi_1}\psi\right) + \frac{\psi_1'}{\psi_1}\psi' + \left(\frac{\psi_1'}{\psi_1}\right)^2\psi \\
&= -\psi'' - \frac{\psi_1''}{\psi_1}\psi + \left(\frac{\psi_1'}{\psi_1}\right)^2\psi - \frac{\psi_1'}{\psi_1}\psi + \frac{\psi_1'}{\psi_1}\psi + \frac{\psi_1''}{\psi_1}\psi + \left(\frac{\psi_1'}{\psi_1}\right)^2\psi \\
&= -\psi'' - \frac{\psi_1''}{\psi_1}\psi + 2\left(\frac{\psi_1'}{\psi_1}\right)^2\psi.
\end{aligned}$$

By rearranging the equation $-\psi_1'' + u\psi_1 = \lambda_1\psi_1$, we obtain

$$\frac{\psi_1''}{\psi_1} = u - \lambda_1.$$

Hence, we get

$$QQ^*\psi = -\psi'' - (u - \lambda_1)\psi + 2\left(\frac{\psi_1'}{\psi_1}\right)^2\psi = \left(\lambda - \lambda_1 + \left(\frac{\psi_1'}{\psi_1}\right)^2\right)\psi$$

Similarly we compute Q^*Q:

$$Q^*Q\psi = \left(-\frac{d}{dx} - \frac{\psi_1'}{\psi_1}\right)\left(\frac{d}{dx} - \frac{\psi_1'}{\psi_1}\right)\psi = -\psi'' - \frac{\psi_1'}{\psi_1}\psi' + \frac{d}{dx}\left(\frac{\psi_1'}{\psi_1}\psi\right) + \left(\frac{\psi_1'}{\psi_1}\right)^2\psi$$
$$= -\psi'' - \frac{\psi_1'}{\psi_1}\psi' + \frac{\psi_1''}{\psi_1}\psi + \frac{\psi_1'}{\psi_1}\psi' - \left(\frac{\psi_1'}{\psi_1}\right)^2\psi + \left(\frac{\psi_1'}{\psi_1}\right)^2\psi.$$

Using again that $\frac{\psi_1''}{\psi_1} = u - \lambda_1$, we obtain that

$$Q^*Q\psi = -\psi'' + (u - \lambda_1)\psi = (\lambda - \lambda_1)\psi.$$

4 the Harmonic Oscillator

Let us consider now the harmonic oscillator which satisfies the equation:

$$-\psi'' + x^2\psi = \lambda\psi.$$

Notice that the function $\psi_1 = e^{-\frac{x^2}{2}}$ is an eigenfunction for the corresponding eigenvalue $\lambda_1 = 1$. Therefore the creation and annihilation operators become:

$$Q = \frac{d}{dx} + x \text{ and } Q^* = -\frac{d}{dx} + x.$$

Similarly we obtain that

$$QQ^*\psi = \left(\frac{d}{dx} + x\right)\left(-\frac{d}{dx} + x\right)\psi = -\psi'' + (x^2+1)\psi \Rightarrow QQ^*\psi = (\lambda+1)\psi$$

and

$$Q^*Q\psi = \left(-\frac{d}{dx} + x\right)\left(\frac{d}{dx} + x\right)\psi = -\psi'' + (x^2-1)\psi \Rightarrow QQ^*\psi = (\lambda-1)\psi.$$

An important property of these operators is that QQ^* and Q^*Q have the same eigenvalues.
Let us consider the function ψ which satisfies the Schrödinger's equation with the eigenvalue λ. We have $Q^*Q\psi = \alpha\psi$, where α, the eigenvalue of the operators, depends on the eigenvalue (of the differential equation) λ. For example, in the case of the harmonic oscillator, $\alpha = \lambda - 1$. If we apply the operator Q to this equation, we obtain

$$Q(Q^*Q)\psi = \alpha Q\psi \Longleftrightarrow (QQ^*)(Q\psi) = \alpha(Q\psi).$$

If we denote $Q\psi = \phi$, we obtain the following equation that $QQ^*\phi = \alpha\phi$. Hence α is also an eigenvalue of the operator QQ^*. This implies that Q^*Q and QQ^* have the same eigenvalues.

4.1 Spectrum of Harmonic Oscillator

From the above discussion, the creation and annihilation operators Q and Q^* satisfy

$$\begin{aligned} Q^*Q\psi &= -\frac{d^2}{dx^2}\psi + x^2\psi - \psi \\ &= H\psi - \mathbb{1}\psi \end{aligned}$$

and

$$\begin{aligned} QQ^*\psi &= -\frac{d^2}{dx^2}\psi + x^2\psi + \psi \\ &= H\psi + \mathbb{1}\psi \end{aligned}$$

where $\mathbb{1}$ is the identity operator for all ψ in the Hilbert space $L^2(\mathbb{R})$ as defined in section 1 and $H = -\frac{d^2}{dx^2} + x^2$.

This enables one to compute the discrete spectrum of the Hamiltonian operator H by computing the discrete spectrum of the new operator $N = Q^*Q$ through

$$\sigma_{disc}(H) = \sigma_{disc}(N) + 1$$

This follows simply from the fact that if $\lambda \in \sigma_{disc}(H)$, then there exists a non-trivial (henceforth, all eigenfunctions are assumed non-trivial, that is no-zero) $\psi \in L^2(\mathbb{R})$ such that $N\psi = \lambda\psi = H\psi - \psi$, giving ψ such that $H\psi = (\lambda + 1)\psi$. this yields the inclusion $\sigma_{disc}(H) \subseteq \sigma_{disc}(N) + 1$. The reverse inclusion is similarly obtained, thereby leading to the desired equality.

Now we consider the spectrum of N. In that direction, as per [16], we make the observation that the spectrum of N must be non-negative, that is $\sigma_{disc}(N) \subseteq \mathbb{R}_{\geq 0}$. This follows by considering $\lambda \in \sigma_{disc}(N)$. This implies the existence of a non-trivial ψ such that $N\psi = Q^*Q\psi = \lambda\psi$. We notice that multiplying both sides of the previous equation by ψ gives

$$\psi Q^*Q\psi = \lambda\psi^2$$

Integrating over $\mathbb{R}$ yields

$$\int_{\mathbb{R}} \psi Q^*Q\psi dx = \lambda \int_{\mathbb{R}} \psi^2 dx$$

$$\|Q\psi\|^2_{L^2(\mathbb{R})} = \lambda \|\psi\|^2_{L^2(\mathbb{R})}$$

hence, $\lambda = \frac{\|Q\psi\|^2_{L^2(\mathbb{R})}}{\|\psi\|^2_{L^2(\mathbb{R})}} \geq 0$, since ψ is non-trivial and using the fact that $\|.\|^2_{L^2(\mathbb{R})}$ is a norm on $L^2(\mathbb{R})$.

Furthermore, we consider the following operators $[N, Q^*] := NQ^* - Q^*N$ and $[N, Q] := NQ - QN$. More explicitly, we compute

$$[N, Q^*] = \left(-\frac{d^2}{dx^2} + x^2 - \mathbb{1}\right)\left(-\frac{d}{dx} + x\right) - \left(-\frac{d}{dx} + x\right)\left(-\frac{d^2}{dx^2} + x^2 - \mathbb{1}\right)$$

and

$$[N, Q] = \left(-\frac{d^2}{dx^2} + x^2 - \mathbb{1}\right)\left(\frac{d}{dx} + x\right) - \left(\frac{d}{dx} + x\right)\left(-\frac{d^2}{dx^2} + x^2 - \mathbb{1}\right)$$

An easy computation yields that $[N, Q^*] = 2Q^*$ and $[N, Q] = -2Q$.

Now suppose $\lambda \in \sigma_{disc}(N)$ and ψ a corresponding eigenfunction. We claim that $Q^*\psi$ is an non-trivial eigenfunction with eigenvalue $\lambda + 2$. The fact that $Q^*\psi$ is non-trivial follows from

$$\begin{aligned}\|Q^*\psi\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \psi QQ^*\psi dx &= \int_{\mathbb{R}} \psi Q^*Q\psi dx + 2\int_{\mathbb{R}} \psi^2 dx \\ &= \|Q\psi\|^2_{L^2(\mathbb{R})} + 2\|\psi\|^2_{L^2(\mathbb{R})} \geq 2\|\psi\|^2_{L^2(\mathbb{R})} > 0\end{aligned}$$

.
Also,

$$NQ^*\psi = [N, Q^*]\psi + Q^*N\psi = 2Q^*\psi + \lambda Q^*\psi = (\lambda + 2)Q^*\psi$$

Additionally, for $\lambda \in \sigma_{disc}(N), \lambda \geq 2$ and ψ a corresponding non-trivial eigenfunction, $Q\psi$ is an non-trivial eigenfunction with eigenvalue $\lambda - 2$. The fact that $Q\psi$ is non-trivial follows from

$$\|Q\psi\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \psi Q^*Q\psi dx = \int_{\mathbb{R}} \psi N\psi dx = \int_{\mathbb{R}} \lambda\psi^2 dx = \lambda\|\psi\|^2_{L^2(\mathbb{R})} > 0$$

.
Also,

$$NQ\psi = [N, Q]\psi + QN\psi = -2Q\psi + \lambda Q\psi = (\lambda - 2)Q\psi$$

Thus, $\lambda \in \sigma_{disc}(N)$ implies $(\lambda + 2) \in \sigma_{disc}(N)$ and $(\lambda - 2) \in \sigma_{disc}(N)$ if $\lambda \geq 2$. We claim that

$$\sigma_{disc}(N) = 2\mathbb{N} \cup \{0\}$$

Consider $\lambda \in \sigma_{disc}(N)$, then λ has the unique decomposition $\lambda = 2n + r, n \in \mathbb{N}$ and $r \in [0, 2)$. If we assume that λ is not an even non-negative integer, then r is strictly positive (an assumption made below). By the above, a repeated application of Q to ψ $n + 1$ times gives the non-trivial $Q^{n+1}\psi$ with eigenvalue $r - 2 < 0$, in contradiction to the non-negativity of the spectrum of N. Thus, $\sigma_{disc}(N) \subseteq 2\mathbb{N} \cup \{0\}$. The reverse inclusion is obtained by considering the sequence of eigenfunctions

$$\left((Q^*)^n\psi_1\right)_{n\in\mathbb{N}\cup\{0\}}$$

with eigenvalues $\lambda_n = 2n$ respectively; where $\psi_1 = \exp(-\frac{x^2}{2}) \in L^2(\mathbb{R})$ is an eigenvector of N, with eigenvalue $\lambda = 0$. Thus,

$$\sigma_{disc}(H) = \sigma_{disc}(N) + 1 = \{1, 3, 5, 7, \dots\}$$

that is, the odd positive integers.

4.2 Solution

As we now know the eigenvalues take the form $2n + 1$, we go about finding corresponding eigenfunctions that solve the equation. We understand that repeated applications of Q^* yield eigenfunctions with increased eigenvalues, and repeated applications of Q lower the eigenvalues. We proceed as follows:
Consider the ground state solution ψ_0 to:

$$-\psi'' + x^2\psi = \lambda\psi$$

such that $Q\psi_0 = 0$. This means that $(\frac{d}{dx} + x)\psi_0 = 0 \implies \frac{d\psi_0}{dx} + x\psi_0 = 0$. Therefore we have $\frac{1}{\psi_0}d\psi_0 = -xdx$, easily yielding an initial solution $\psi_0 = Ce^{\frac{-x^2}{2}}$, from which we take $e^{\frac{-x^2}{2}}$ as our initial solution.
Reinjecting this solution into the original differential operator $(\frac{d^2}{dx^2} + x^2)$ yields

$$-\frac{d^2}{dx^2}e^{\frac{-x^2}{2}} + x^2e^{\frac{-x^2}{2}} = e^{\frac{-x^2}{2}}$$

Thus, we have confirmed that our initial solution also gives rise to the lowest eigenvalue in our sequence, such that $\lambda_0 = 1$.
As mentioned earlier, finding the remaining eigenfunctions now reduces to applying Q^* repeatedly to our initial solution.

$$Q^*e^{\frac{-x^2}{2}} = (-\frac{d}{dx} + x)e^{\frac{-x^2}{2}} = 2xe^{\frac{-x^2}{2}} = \psi_1$$

This, when reinjected, also produces an eigenvalue $\lambda_1 = 3$, adhering to the expected sequence. One more iteration yields

$$\psi_2 = Q^*\psi_1 = (x - \frac{d}{dx})2xe^{\frac{-x^2}{2}} = (4x^2 - 2)e^{\frac{-x^2}{2}} = (4x^2 - 2)\psi_0$$

A straightforward reinjection shows that this solution ψ_2 has an eigenvalue $\lambda_2 = 5$. Additionally, a pattern begins to emerge where our solutions take the form of a polynomial multiplied by our initial Gaussian function ψ_0. This is confirmed when we consider arbitrary many iterations:

$$\psi_n = (Q^*)^n\,\psi_0 = (x - \frac{d}{dx})^n\psi_0$$

We proceed with an inductive argument to find ψ_n in general. Firstly, notice that ψ_0 corresponds to $H_0(x)e^{-\frac{x^2}{2}}$, where $H_n(x)$ denotes the nth Hermite polynomial. Similarly, our computations for ψ_1 and ψ_2 yield solutions $H_1(x)e^{-\frac{x^2}{2}}$

and $H_2(x)e^{-\frac{x^2}{2}}$, respectively, suggesting that solutions take the form of a Hermite polynomial multiplied by $e^{-\frac{x^2}{2}}$.
Suppose that for some n, $\psi_n = H_n(x)e^{-\frac{x^2}{2}}$. We therefore have that

$$\psi_{n+1} = Q^*\psi_n = (x - \frac{d}{dx})H_n(x)e^{-\frac{x^2}{2}}$$

$$= xH_n(x)e^{-\frac{x^2}{2}} - H_n'(x)e^{-\frac{x^2}{2}} + xH_n(x)e^{-\frac{x^2}{2}}$$

$$= (2xH_n(x) - H_n'(x))e^{-\frac{x^2}{2}} = H_{n+1}(x)e^{-\frac{x^2}{2}}$$

where in the final equality we have used the fact that the recurrence relation for Hermite polynomials defines $H_{n+1}(x) = 2xH_n(x) - H_n'(x)$. Thus, as ψ_0 satisfies the base case, the induction on n for ψ_n is complete.
All that is left to check is that ψ_n does indeed have eigenvalue $\lambda_n = 2n+1$. We proceed by injecting our general solution into the Hamiltonian and solving for λ:

$$(-\frac{d^2}{dx^2} + x^2)\psi_n = (-\frac{d^2}{dx^2} + x^2)H_n(x)e^{-\frac{x^2}{2}} = -\frac{d^2}{dx^2}[H_n(x)e^{-\frac{x^2}{2}}] + x^2H_n(x)e^{-\frac{x^2}{2}}$$

yielding

$$(H_n(x) + 2xH_n'(x) - H_n''(x))e^{-\frac{x^2}{2}}$$

We use the following property of Hermite polynomials to simplify:

$$H_n'(x) = 2nH_{n-1}(x)$$

Which when applied to the expression yields

$$(H_n(x) + 4xnH_{n-1}(x) - 4n^2H_{n-2}(x))e^{-\frac{x^2}{2}} = (H_n(x) + 2n(2xH_{n-1}(x) - 2nH_{n-2}(x)))e^{-\frac{x^2}{2}}$$

Finally, we use the recurrence relation for Hermite polynomials: $H_n(x) = 2xH_{n-1}(x) - 2nH_{n-2}(x)$:

$$(H_n(x) + 2n(2xH_{n-1}(x) - 2nH_{n-2}(x)))e^{-\frac{x^2}{2}} = (2n+1)H_n(x)e^{-\frac{x^2}{2}} = (2n+1)\psi_n$$

Thus confirming that each ψ_n takes eigenvalue $\lambda_n = 2n + 1$, as expected.

5 Pöschl–Teller Potentials

In this section, we will be considering the Schrödinger operator

$$H = -\frac{d^2}{dx^2} - \ell(\ell + 1)\operatorname{sech}^2(x)$$

and its spectrum.

5.1 Hamiltonian related to Q^*Q and QQ^*

Let's denote the Hamiltonian operator with respect to ℓ by H_ℓ, considering the one-dimensional Schrödinger equation with a Pöschl-Teller potential:

$$H_\ell \psi = -\frac{d^2}{dx^2}\psi - \frac{\ell(\ell+1)}{\cosh^2 x}\psi = \lambda\psi$$

The lowest eigenfunction (eigenfunction corresponds to the most negative eigenvalue) is

$$\psi_1 = \frac{1}{\cosh^\ell x}$$

Therefore, the creation and annihilation operators become:

$$\begin{aligned}
Q = \frac{d}{dx} - \frac{\psi_1'}{\psi_1} = \frac{d}{dx} - \frac{\frac{d}{dx}(\operatorname{sech}^\ell x)}{\operatorname{sech}^\ell x} &= \frac{d}{dx} - \frac{\ell \operatorname{sech}^{\ell-1} x(-\operatorname{sech} x \tanh x)}{\operatorname{sech}^\ell x} \\
&= \frac{d}{dx} - \frac{\ell \tanh x \operatorname{sech}^\ell x}{\operatorname{sech}^\ell x} = \frac{d}{dx} - \ell \tanh x \\
Q^* = -\frac{d}{dx} - \frac{\psi_1'}{\psi_1} = -\frac{d}{dx} - \frac{\frac{d}{dx}(\operatorname{sech}^\ell x)}{\operatorname{sech}^\ell x} &= -\frac{d}{dx} - \frac{\ell \operatorname{sech}^{\ell-1} x(-\operatorname{sech} x \tanh x)}{\operatorname{sech}^\ell x} \\
&= -\frac{d}{dx} - \frac{\ell \tanh x \operatorname{sech}^\ell x}{\operatorname{sech}^\ell x} = -\frac{d}{dx} - \ell \tanh x
\end{aligned}$$

[7]

We compute the values of Q^*Q and QQ^*:

$$\begin{aligned}
Q^*Q = \left(\frac{d}{dx} - \ell \tanh x\right)\left(-\frac{d}{dx} - \ell \tanh x\right) &= -\frac{d^2}{dx^2} - \ell \operatorname{sech}^2 x + \ell^2 \tanh^2 x \\
&= -\frac{d^2}{dx^2} - \ell \operatorname{sech}^2 x + \ell^2(1 - \operatorname{sech}^2 x) \\
&= -\frac{d^2}{dx^2} + \ell^2 - \ell(\ell+1) \operatorname{sech}^2 x \\
Q^*Q = \left(-\frac{d}{dx} - \ell \tanh x\right)\left(\frac{d}{dx} - \ell \tanh x\right) &= -\frac{d^2}{dx^2} + \ell \operatorname{sech}^2 x + \ell^2 \tanh^2 x \\
&= -\frac{d^2}{dx^2} + \ell \operatorname{sech}^2 x + \ell^2(1 - \operatorname{sech}^2 x) \\
&= -\frac{d^2}{dx^2} + \ell^2 - \ell(\ell-1) \operatorname{sech}^2 x
\end{aligned}$$

Since we know

$$\begin{aligned}
H_\ell &= -\frac{d^2}{dx^2} - \frac{\ell(\ell+1)}{\cosh^2 x} = -\frac{d^2}{dx^2} - \ell(\ell+1) \operatorname{sech}^2 x \\
H_{\ell-1} &= -\frac{d^2}{dx^2} - \frac{\ell(\ell-1)}{\cosh^2 x} = -\frac{d^2}{dx^2} - \ell(\ell-1) \operatorname{sech}^2 x
\end{aligned}$$

The following relations of H, Q^*Q and QQ^* can be deduced:

$$Q^*Q = H_\ell + \ell^2$$

$$QQ^* = H_{\ell-1} + \ell^2$$

5.2 Spectrum of the Hamiltonian

As seen in the previous section we have found a relation between H_ℓ, $H_{\ell+1}$ and the operators Q_ℓ and Q_ℓ^*:

$$\begin{aligned} Q^*Q &= H_\ell + \ell^2 \\ QQ^* &= H_{\ell-1} + \ell^2. \end{aligned}$$

By using the property proved in section 3, that QQ^* and Q^*Q have the same eigenvalues. We can conclude that the spectrum of H_ℓ coincides with the spectrum of $H_{\ell+1}$ except for the eigenvalue corresponding to the ground state. In other words, ℓ eigenvalues are coming from the eigenvalues of the H_ℓ operator to which we apply Q_{l+1}^* and one eigenvalue yields from the ground state, which means $Q_{\ell+1}\psi = 0$. We have seen in the previous sections that the eigenvalue for the bounded solution for $\ell = 1$ is $\lambda = -1$. Similarly for the case $\ell = 2$, the eigenvalues are $\lambda = -1, -4$. Therefore, by induction, for fixed ℓ, the eigenvalues for the bounded solutions are $\lambda = -1, -2^2, \ldots, -\ell^2$.

5.3 $\ell = 0$: Free Hamiltonian

In this case, $H = -\frac{d^2}{dx^2}$; it corresponds to the 'free' particle in Quantum Mechanics, that is a particle under the influence of no external forces. One immediately notices that the spectrum for this operator is likely to be radically different than that of the harmonic oscillator due to the qualitatively different asymptotics of the potentials in question. This can be seen by considering for $k \in \mathbb{R}$ arbitrary, the function $\psi_k = \exp(ikx)$. It is plain that $H\psi_k = k^2\psi_k$. But, $\psi_k \notin L^2(\mathbb{R})$.

This necessitates the broadening of the definition of the 'spectrum' of an operator that we have been suppressing in favour of the more naive definition for linear operators on finite dimensional vector spaces. Using the theory developed in previous sections, this can be made precise. We claim that such functions are 'generalised eigenvectors' in the sense that their eigenvalues belong to the essential spectrum of H. In other words, we claim that $[0, \infty) \subseteq \sigma_{ess}(H)$.

We begin, following [5], by constructing a suitable singular sequence for $k \in [0, \infty)$. Let $w(x) = \exp(ikx)$ and furthermore, let $\psi(x) \in C_c^\infty(\mathbb{R})$ be the smooth cutoff function that equals to unity inside the interval $B_1 = [-1, 1]$ and zero outside the interval $B_2 = [-2, 2]$. Existence is guaranteed, see [11]. Further define $\psi(x)_n = \psi(\frac{x}{n})$; note it equals to unity inside the interval $B_n = [-n, n]$

and zero outside the interval $B_{2n} = [-2n, 2n]$.

Now, set $\eta_n = \psi_n(x)w(x)$ and $c_n = \frac{1}{||\eta_n||}$. We claim that $u_n = c_n\eta_n$ is the desired singular sequence.

The fact that $||u_n|| = 1$ is clear; to show that $w - \lim\limits_{n\to\infty} u_n = 0$, it suffices to consider $\langle f, u_n\rangle$ for $f \in C_c^\infty(\mathbb{R})$ by density in $L^2(\mathbb{R})$. This follows from the Cauchy-Schwartz inequality obeyed by the inner product on $H = L^2(\mathbb{R})$ as

$$|\langle f - f_n, u\rangle| \leq ||f - f_n|.|||u|| \to 0$$

for a sequence $f_n \in C_c^\infty(\mathbb{R})$ converging strongly to $f \in L^2(\mathbb{R})$. Fix $f \in C_c^\infty(\mathbb{R})$ and consider the inner product

$$|\langle f, u_n\rangle| = \left|\int_{B_{2n}} f\frac{\eta_n}{||\eta_n||}dx\right|$$

$$\leq \left|\int_{B_n} f\frac{1}{||\eta_n||}dx\right| + \left|\int_{B_{2n}\setminus B_n} f\frac{\eta_n}{||\eta_n||}dx\right|$$

$$\leq \frac{||f||_1}{||\eta_n||} + \left|\int_{B_{2n}\setminus B_n} f\frac{\eta_n}{||\eta_n||}dx\right| \to 0$$

as $||\eta_n|| \geq \sqrt{\left|\int_{B_n} dx\right|} = \sqrt{2n}$ and due to the compactness of the support of f,

$$\left|\int_{B_{2n}\setminus B_n} f\frac{\eta_n}{||\eta_n||}dx\right| = 0$$

for sufficiently large n such that $\text{supp}(f) \subseteq B_n$.

To finish the proof of the claim, it suffices to show that

$$||Hu_n - ku_n|| \to 0$$

as $n \to \infty$. Now,

$$||Hu_n - ku_n||^2 = \int_{\mathbb{R}} \left|\left(-\frac{d^2}{dx^2} - k\right)\frac{\eta_n}{||\eta_n||}\right|^2 dx$$

$$= \frac{1}{||\eta_n||^2}\left[\int_{B_n}\left|\left(-\frac{d^2}{dx^2} - k\right)w(x)\right|^2 dx + \int_{B_{2n}\setminus B_n}\left|\left(-\frac{d^2}{dx^2} - k\right)\psi_n w(x)\right|^2 dx\right]$$

The first integral vanishes due to $w(x) = \exp(ikx)$ which plainly yields that $\left|\left(-\frac{d^2}{dx^2} - k\right) w(x)\right| = 0, \forall x \in \mathbb{R}$. Thus,

$$||Hu_n - ku_n||^2 = \frac{1}{||\eta_n||^2} \left[\int_{B_{2n}\setminus B_n} \left| \left(-\frac{d^2}{dx^2} - k \right) \psi_n w(x) \right|^2 dx \right]$$

$$= \frac{1}{||\eta_n||^2} \left[\int_{B_{2n}\setminus B_n} \left| \left(-\frac{d^2\psi_n}{dx^2} w(x) - 2\frac{d\psi_n}{dx}\frac{dw(x)}{dx} - \frac{d^2 w(x)}{dx^2}\psi_n(x) - k\psi_n w(x) \right) \right|^2 dx \right]$$

$$= \frac{1}{||\eta_n||^2} \left[\int_{B_{2n}\setminus B_n} \left| \left(-\frac{d^2\psi_n}{dx^2} w(x) - 2\frac{d\psi_n}{dx}\frac{dw(x)}{dx} \right) \right|^2 dx \right]$$

by the product rule for one-dimensional functions and again that $\left|\left(-\frac{d^2}{dx^2} - k\right) w(x)\right| = 0, \forall x \in \mathbb{R}$. Using the Cauchy-Schwarz inequality for the inner product, we get

$$||Hu_n - ku_n||^2 \leq \frac{2}{||\eta_n||^2} \left[\int_{B_{2n}\setminus B_n} \left| \frac{d^2\psi_n}{dx^2} w(x) \right|^2 dx + \int_{B_{2n}\setminus B_n} \left| 2\frac{d\psi_n}{dx}\frac{dw(x)}{dx} \right|^2 dx \right]$$

$$= \frac{2}{||\eta_n||^2} \left[\int_{B_{2n}\setminus B_n} \left| \frac{d^2\psi_n}{dx^2} \right|^2 dx + \int_{B_{2n}\setminus B_n} \left| 2\frac{d\psi_n}{dx}\frac{dw(x)}{dx} \right|^2 dx \right]$$

$$= \frac{2}{||\eta_n||^2} \left[\int_{B_{2n}\setminus B_n} \left| \frac{d^2\psi_n}{dx^2} \right|^2 dx + 4k^2 \int_{B_{2n}\setminus B_n} \left| \frac{d\psi_n}{dx} \right|^2 dx \right]$$

The fact that $\psi(x) \in C_c^\infty(\mathbb{R})$ means that $\frac{d\psi(x)}{dx}, \frac{d^2\psi(x)}{dx^2}$ are continuous and compactly supported, thus bounded by some $M < \infty$ in absolute value. Thus, in conjunction with the chain rule, one obtains

$$\left| \frac{d\psi(x)}{dx} \right| \leq \frac{M}{n}, \left| \frac{d^2\psi(x)}{dx^2} \right| \leq \frac{M}{n^2}$$

This enables us to give the following bound

$$||Hu_n - ku_n||^2 \leq \frac{2}{2n} \left[2n\frac{M^2}{n^4} + 4k^2 2n \frac{M^2}{n^4} \right]$$

$$= 2\left[\frac{M^2}{n^4} + 4k^2 \frac{M^2}{n^4} \right] \to 0$$

as $n \to \infty$ completing the proof of the claim.

5.4 $\ell = 1$:

For the case $\ell = 1$, the equation is

$$-\psi'' - \frac{2}{\cosh^2(x)}\psi = \lambda\psi \tag{1}$$

and our Hamiltonian is $H_1 = -\frac{d^2}{dx^2} - \frac{2}{\cosh^2(x)}$.
For this equation, the creation and annihilation operators are as below:

$$Q_1 = \frac{d}{dx} - \frac{\psi_1'}{\psi_1} = \frac{d}{dx} - \frac{-\sinh(x)\operatorname{sech}^2(x)}{\operatorname{sech}(x)} = \frac{d}{dx} + \tanh(x) \tag{2}$$

$$Q_1^* = -\frac{d}{dx} - \frac{\psi_1'}{\psi_1} = -\frac{d}{dx} - \frac{-\sinh(x)\operatorname{sech}^2(x)}{\operatorname{sech}(x)} = -\frac{d}{dx} + \tanh(x) \tag{3}$$

[7]
According to the previous parts, and according to [4] the spectrum of H_1 is the same as that of H_0 except for the 0 eigenvalue. Thus, we can see that there exists a continuum of eigenvalue and eigenfunctions for positive eigenvalues.[4]
When $(Q_1^*Q_1)\psi = 0$, it is another eigenfunction of H_1. Since we have $H_1 = Q_1^*Q_1 + \ell^2 = Q_1^*Q_1 + 1$, and

$$(Q_1^*Q_1)\psi = 0 \Rightarrow (H_1 + 1)\psi = 0 \Rightarrow H_1\psi = -\psi$$

such ψ is an eigenfunction with eigenvalue -1. And it satisfies the differential equation

$$(\frac{d}{dx} + tanhx)\psi = 0$$

hence we have $\psi_1^{(0)} = N_1 sechx$.[4]
And the solutions corresponding to the continuum can be obtained by applying Q_1^* to the continuum eigenfunctions of H_0. That is,

$$Q_1^* \exp(ikx) = (-\frac{d}{dx} + tanhx)\exp(ikx) = \exp(ikx)(ik + tanhx)$$

5.5 $\ell = n, n \in \mathbb{N}$

Similar to the previous case, for a $\ell = n$, there is a continuum for positive eigenvalues. And there is going to be n negative eigenvalues for H_n.[4]
The n eigen values are $\{-j^2 \mid 1 \leq j \leq n\}$. When $(Q_1^*Q_1)\psi = 0$, we get the eigenfunction with eigenvalue $-n^2$, similar to the case for $\ell = 1$,

$$(Q_n^*Q_n)\psi = 0 \Rightarrow (H_n + n^2)\psi = 0 \Rightarrow H_1\psi = -n^2\psi$$

And such a function is the solution of the differential equation

$$\left(\frac{d}{dx} + n \cdot \mathrm{tanhx}\right)\psi = 0$$

that is, $N_n \operatorname{sech}^n x$.[4]
While the eigenfunctions and eigenvalues $\{-j^2 \mid 1 \leq j \leq (n-1)\}$ are obtained by applying Q^* to the eigenfunction with eigenvalue $-j^2$ of operator H_j for cases that $\ell = j$ for $j \in \{1, 2, ..., n-1\}$, which have the form

$$(\frac{d}{dx} + n\tanh(x))N_j \operatorname{sech}^j(x) = -N_j j \sinh(x) \operatorname{sech}^{j+1}(x) + nN_j \tanh(x) \operatorname{sech}^j(x)$$

The eigenfunctions for the continuum can be obtained by the same method as in the case for $\ell = 1$: by repeatedly applying Q_n^* to their counterparts for the equation with $\ell = n - 1$. That is,

$$(-\frac{d}{dx} + n\tanh(x))^n \exp(ikx)$$

5.6 Eigenstates for general ℓ: Legendre functions

Section 5.5 outlines a general method for now finding all solutions for integer values of ℓ. Recall that relationships between Q^*Q and $QQ*$ imply that the Hamiltonians with respect to each ℓ have the same spectrum, with the exception of the ground state case. This also holds true for the essential spectrum $\sigma_{\text{ess}} = [0, \infty)$. In the $\ell = 1$ case we have already computed the eigenfunctions as being $\exp(ikx)$, and in section 5.3 we have already shown the solution for the essential spectrum, which is not technically admissible as it lies outside of $L^2(\mathbb{R})$.
Calculating the bound state solutions, on the other hand, is rather straightforward. As we know that for each ℓ, $\sigma_{\text{disc}}(H_\ell) \supset \sigma_{\text{disc}}(H_{\ell-1})$. Where the proper superset property is provided based off of the fact that $\sigma_{\text{disc}}(H_\ell)$ contains exactly one more element than $\sigma_{\text{disc}}(H_{\ell-1})$, namely when $Q_\ell = 0$. In this sense, the solutions follow a recursive pattern, where for any $\ell + 1$, ℓ solutions are provided by raising existing solutions using the creation operator Q_ℓ^*, and 1 additional solution is provided by calculating the ground state for each ℓ, such that $Q\psi_1^{(\ell+1)} = 0$. Furthermore, the eigenvalue of each state is preserved (by definition), and the energy value of each ground state is given by:

$$Q_\ell \psi_1^{(\ell)} = 0 \implies [\frac{d}{dx} + \ell \tanh x]\psi_1^{(\ell)} = 0 \implies \psi_1^{(\ell)} = \operatorname{sech}^\ell x$$

A straightforward reinjection shows that the ground state eigenvalue is equal to $-\ell^2$, and so we deduce that the eigenvalues for each ℓ are exactly the set $\sigma_\ell = \{-m^2 | m \in \mathbb{N}, m \leq \ell\}$
We consider now the general form of the eigenfunctions for arbitrary ℓ, each of which will be denoted by $\psi_n^{(\ell)}$, where n indexes each eigenfunction in terms of ascending eigenvalues. Clearly, as the eigenvalues are a decreasing sequence, we wish to move backwards - that is, we begin with the ground state for ℓ, then

the eigenfunction inherited from the ground state of $\ell-1$, and so on.
The sequence is thus as follows:

$$\psi_1^{(\ell)},\ \psi_2^{(\ell)} = Q_\ell^*\psi_1^{(\ell-1)},\ \psi_3^{(\ell)} = Q_\ell^*\psi_2^{(\ell-1)} = Q_\ell^* Q_{\ell-1}^*\psi_1^{(\ell-2)}$$

And in general, a pattern forms where the nth solution can be reduced to:

$$\psi_n^{(\ell)} = Q_\ell^* Q_{\ell-1}^* \ldots Q_{\ell-n+2}^*\psi_1^{(\ell-n+1)}$$

Which can be further expanded to:

$$\psi_n^{(\ell)} = (-\frac{d}{dx}+\ell\tanh x)(-\frac{d}{dx}+(\ell-1)\tanh x)\ldots(-\frac{d}{dx}+(\ell-n+2)\tanh x)\psi_1^{(\ell-n+1)}$$

And so we have devised a system for calculating the bound states of the system through application of computationally simple operators to known ground states. The derivation of the closed form solution, as associated Legendre polynomials, however, requires more calculus which will be addressed further on.

5.7 Non-reflectivity

- explicit calculation for case $\ell = 1$
  As seen in the previous part, we find the explicit solution for $\ell = 1$ from $\psi_1(x) = Q_\ell^*\psi_1$, which gives us that

  $$\psi_1(k, x) = (k + i\tanh x)\exp(ikx).$$

  To observe the continuum states, we apply the usual parametrization of the form

  $$\lim_{x\to-\infty}\psi(k, x) = e^{ikx} + R(k)e^{-ikx}$$
  $$\lim_{x\to\infty}\psi(k, x) = T(k)e^{ikx},$$

  where the function $R(k)$ is the reflection coefficient and $T(k)$ is the transmission coefficient. Applying these to the solution ψ_1, we obtain:

  $$\lim_{x\to-\infty}\psi_1(k, x) = (k - i)e^{ikx}$$
  $$\lim_{x\to\infty}\psi(k, x) = (k + i)e^{ikx},$$

  hence we obtain that the reflection coefficient is $R(k) = 0$, in other words the equation is reflectionless, and the transmission coefficient is $T(k) = \frac{k+i}{k-i}$ (or $T(k) = \exp{(2i)}\tan^{-1}(\frac{1}{k})$).
  For the case $l = 2$, we obtain

  $$\psi_2 = Q_2^*\psi_1 = (-i\frac{d}{dx}+2i\tanh x)(k+i\tanh x)e^{ikx} = (1+k^2+3i\tanh x-3\tanh^2 x)e^{ikx}.$$

Proceeding with the same computations, we obtain that $R(k) = 0$, which implies that the equation is reflectionless and

$$T(k) = \frac{(k+i)(k+2i)}{(k-i)(k-2i)} = \exp(2i(\tan^{-1}\left(\frac{1}{k}\right) + \tan^{-1}\left(\frac{2}{k}\right))).$$

- Transmission coefficient
  We apply this argument to the general case for ℓ, aiming to prove that the equation is reflectionless, which in other words means that $R(k) = 0$ and to find a general form of the transmission coefficient $T(k)$.
  From the previous section, we can find that

$$\psi_\ell(k,x) = Q_\ell^* \psi_{\ell-1}(k,x).$$

Hence, if we take the limits to $-\infty$ and ∞, we obtain

$$\begin{aligned} \lim_{x\to-\infty} \psi_\ell(k,x) &= \lim_{x\to-\infty} \left(-i\frac{d}{dx} + \ell i \tanh x\right)\psi_{\ell-1} \\ &= -i\frac{d}{dx}\left(T(\ell-1)e^{ikx}\right) + \ell i \tanh x T(\ell-1)e^{ikx} \\ &= kT(\ell-1) + \ell i T(\ell-1))e^{ikx} = (k+\ell i)T(\ell-1)e^{ikx} \end{aligned}$$

and

$$\begin{aligned} \lim_{x\to\infty} \psi_\ell(k,x) &= \lim_{x\to\infty} \left(-i\frac{d}{dx} + \ell i \tanh x\right)\psi_{\ell-1} \\ &= -i\frac{d}{dx}\left(T(\ell-1)e^{ikx}\right) + \ell i \tanh x T(\ell-1)e^{ikx} \\ &= (kT(\ell-1) - \ell i T(\ell-1))e^{ikx} = (k-\ell i)T(\ell-1)e^{ikx}. \end{aligned}$$

Therefore, we can conclude that the transmission coefficient is $T(\ell) = \frac{k+\ell i}{k-\ell i}T(\ell-1)$, which implies that

$$T(\ell) = \frac{(k+i)(k+2i)\dots(k+\ell i)}{(k-i)(k-2i)\dots(k-\ell i)} = \exp\left(2i\sum_{j=1}^{\ell}\tan^{-1}(\frac{j}{\ell})\right).$$

On the other hand, we obtain that $R(k) = 0$, which again implies that the equation is reflectionless. In the context of the physical world, the solution is a travelling wave to the right with no reflected wave. Also notice that

$$\lim_{x\to\infty} |\psi_\ell(x)|^2 = \lim_{x\to-\infty} |\psi_\ell(x)|^2,$$

which implies that any particle coming from the left side will pass straight through to the right with no reflection.

5.8 Alternative Solution to the Poschl-Teller Potential

There is an alternative approach towards solving the Poschl-Teller potential without usage of the Creation and Annihilation operators, and it yields solutions in closed form, which is not straightforward using the Creation and Annihilation operators. Consider once again Schrodinger's equation with a Poschl-Teller Potential:

$$-\psi'' - \frac{\ell(\ell+1)}{\cosh^2 x}\psi = \lambda\psi$$

And the substitution $u = \tanh x$ [1]. From this substitution we consider now the following associated equalities:

$$x = \tanh^{-1} u,\ \frac{1}{\cosh^2 \tanh^{-1} x} = 1 - x^2,\ \frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2}$$

Therefore we now have:

$$\psi'' = \frac{d}{dx}\left[\frac{d\psi}{dx}\right] = (1-u^2)\frac{d}{du}\left[(1-u^2)\frac{d\psi}{du}\right] = (1-u^2)^2\frac{d^2\psi}{du^2} + (1-u^2)(-2u\frac{d\psi}{du})$$

And so the entire Poschl-Teller equation can be rewritten as:

$$-(1-u^2)\left[(1-u^2)\frac{d^2\psi}{du^2} - 2u\frac{d\psi}{du}\right] - \ell(\ell+1)(1-u^2)\psi - \lambda\psi = 0$$

Simplification yields

$$(1-u^2)\frac{d^2\psi}{du^2} - 2u\frac{d\psi}{du} + \left[\ell(\ell+1) + \frac{\lambda}{1-u^2}\right]\psi = 0$$

Defining $\lambda = -\mu^2$ finally yields:

$$(1-u^2)\psi'' - 2u\psi' + \left[\ell(\ell+1) - \frac{\mu^2}{1-u^2}\right] = 0$$

And this happens to exactly be the Legendre equation, whose solutions in general are written in the form $\psi(u) = P_\ell^\mu(u)$, where P_ℓ^μ denotes the associated Legendre polynomial, which is admissible in the case where ℓ is an integer, which is assumed to be true.
Thus, our solution reads:

$$\psi(x) = P_\ell^\mu(\tanh x),\ \lambda = -\mu^2$$

And in general, $\mu = 1, 2, 3 \ldots \ell - 1, \ell$.
One major point to notice is that this solution does not independently give way to the same free particle solutions as the prior method, in that the substitution solution only allows for bound state eigenvalues (note that the substitution $\lambda = -\mu^2$ restricts λ exclusively to negative integers).

6 Coulomb potential

6.1 The operator and spectrum of the Hamiltonian

It can be verified that the function $\psi_1 = r^{\ell+1}e^{-\frac{\kappa r}{2(\ell+1)}}$ is a solution for the equation, for some λ.[8]

The Hamiltonian for ℓ is

$$H_\ell = -\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} - \frac{\kappa}{r} \tag{4}$$

As in the previous cases, we define the operators for the Darboux Transformation, for a fixed arbitrary ℓ as below:

$$Q_\ell = \frac{d}{dr} - \frac{\frac{d\psi_1}{dr}}{\psi_1} = \frac{d}{dr} - \frac{\ell+1}{r} + \frac{\kappa}{2(\ell+1)} \tag{5}$$

$$Q_\ell^* = -\frac{d}{dr} - \frac{\frac{d\psi_1}{dr}}{\psi_1} = -\frac{d}{dr} - \frac{\ell+1}{r} + \frac{\kappa}{2(\ell+1)} \tag{6}$$

[8]
Applying (3) and (4) to a function ψ, we can obtain:

$$\begin{aligned}
(Q_\ell^* Q_\ell)\psi &= \left(-\frac{d}{dx} - \frac{\ell+1}{r} + \frac{\kappa}{2(\ell+1)}\right)\left(\frac{d}{dr} - \frac{\ell+1}{r} + \frac{\kappa}{2(\ell+1)}\right)\psi \\
&= -\frac{d^2\psi}{dr^2} + \frac{d}{dr}\left(\frac{\ell+1}{r}\psi\right) - \frac{d}{dr}\left(-\frac{\kappa}{2(\ell+1)}\psi\right) - \frac{\ell+1}{r}\frac{d\psi}{dr} + \frac{(\ell+1)^2}{r^2}\psi - \frac{\kappa}{2r}\psi \\
&+ \frac{\kappa}{2(\ell+1)}\frac{d\psi}{dr} - \frac{\kappa}{2r}\psi + \frac{\kappa^2}{4(\ell+1)^2}\psi \\
&= -\frac{d^2\psi}{dr^2} + \frac{\ell(\ell+1)}{r^2} - \frac{\kappa}{r}\psi + \frac{\kappa^2}{4(\ell+1)^2}\psi
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
(Q_\ell Q_\ell^*)\psi &= \left(\frac{d}{dx} - \frac{\ell+1}{r} + \frac{\kappa}{2(\ell+1)}\right)\left(-\frac{d}{dr} - \frac{\ell+1}{r} + \frac{\kappa}{2(\ell+1)}\right)\psi \\
&= -\frac{d^2\psi}{dr^2} - \frac{d}{dr}\left(\frac{\ell+1}{r}\psi\right) + \frac{d}{dr}\left(-\frac{\kappa}{2(\ell+1)}\psi\right) + \frac{\ell+1}{r}\frac{d\psi}{dr} + \frac{(\ell+1)^2}{r^2}\psi - \frac{\kappa}{2r}\psi \\
&- \frac{\kappa}{2(\ell+1)}\frac{d\psi}{dr} - \frac{\kappa}{2r}\psi + \frac{\kappa^2}{4(\ell+1)^2}\psi \\
&= -\frac{d^2\psi}{dr^2} + \frac{(\ell+1)(\ell+2)}{r^2} - \frac{\kappa}{r}\psi + \frac{\kappa^2}{4(\ell+1)^2}\psi
\end{aligned} \tag{8}$$

Comparing the Hamiltonian H_ℓ, Q_ℓ and Q_ℓ^*, we can establish the following relations:

$$Q_\ell^* Q_\ell = H_\ell + \frac{\kappa^2}{4(\ell+1)} \tag{9}$$

$$Q_\ell Q_\ell^* = H_{\ell+1} + \frac{\kappa^2}{4(\ell+1)} \tag{10}$$

Thus, we can deduce the relation between the spectrum of H_ℓ and $Q_\ell Q_\ell^*$, $Q_\ell^* Q_\ell$:

$$\sigma(H_\ell) = \sigma(Q_\ell^* Q_\ell) - \frac{\kappa^2}{4(\ell+1)^2} \tag{11}$$

and

$$\sigma(H_\ell) = \sigma(Q_{\ell-1} Q_{\ell-1}^*) - \frac{\kappa^2}{4(\ell+1)^2} \tag{12}$$

The kernel of $Q_\ell^* Q_\ell$ are therefore eigenfunctions of H_ℓ with eigenvalue $-\frac{\kappa^2}{4(\ell+1)^2}$, and similarly, the kernel of $Q_\ell Q_\ell^*$ are eigenfunctions of $H_{\ell+1}$ with eigenvalue $-\frac{\kappa^2}{4(\ell+1)^2}$. The spectrum of H_ℓ is equal to the spectrum of $H_{\ell+1}$ apart from the the added eigenvalue, which comes from the ground state of $H_{\ell+1}$ (similar to the case of the Poschl-Teller equation). Out of the $\ell+1$ eigenvalues, ℓ are obtained by applying the operator Q to the previously existing solutions and 1 additional value obtained by calculating the ground state, $Q_{\ell+1}\psi = 0$.

The final Schrödinger operator to be considered is

$$H = -\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} - \frac{\kappa}{r}$$

with $\ell > -\frac{1}{2}, \kappa > 0$ and its discrete spectrum corresponding to the bound, that is normalisable, or in $L^2(\mathbb{R})$.

6.2 Asymptotics of eigen-functions of bound states

Having the goal of computing the discrete spectrum and its corresponding family of eigen functions corresponding to 'bound' states in the physics literature, we shall follow [12] consider the following limiting situations $r \to \infty$ and $r \to 0$ in the eigenvalue problem

$$H\psi = -\frac{d^2\psi}{dr^2} + \frac{\ell(\ell+1)\psi}{r^2} - \frac{\kappa\psi}{r} = \lambda\psi$$

with $\lambda \in (-\infty, 0]$. First, we first consider the limit as $r \to \infty$ and obtain

$$H\psi = O\left(\frac{1}{r}\right)\psi \sim -\frac{d^2}{dr^2}\psi = \lambda\psi$$

which yields

$$\psi \sim \exp(\pm\sqrt{-\lambda}r)$$

as $r \to \infty$ [3]. The solutions with asymptotics $\exp(\sqrt{\lambda}r)$ for large r are discarded as they yield non-normalisable solutions, that is not in $L^2(\mathbb{R})$. Thus, we restrict our attention to normalisable eigenfunctions of the form $\psi = \exp(-\sqrt{\lambda}r)\eta(r)$, that satisfy

$$\left[-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} - \frac{\kappa}{r} - \lambda\right]\psi = 0$$

which, when one substitutes ψ yields

$$\left[-\frac{d^2}{dr^2} + 2\sqrt{-\lambda}\frac{d}{dr} + \frac{\ell(\ell+1)}{r^2} - \frac{\kappa}{r} - \lambda\right]\eta = 0$$

Assuming $\eta \sim r^s$ as $r \to 0$ gives upon substitution to the above

$$s(s-1)r^{s-2} + 2\sqrt{-\lambda}sr^{s-1} + \ell(\ell+1)r^{s-2} - \kappa r^{s-1} - \lambda r^s = 0$$

$$\left[s(s-1)\frac{1}{r^2} + 2\sqrt{-\lambda}s\frac{1}{r} + \ell(\ell+1)\frac{1}{r^2} - \kappa\frac{1}{r} - \lambda\right]r^s = 0$$

Taking $r \to 0$, we neglect terms of order $O\left(\frac{1}{r}\right))$ and obtain

$$s(s-1) = \ell(\ell+1)$$

giving two solutions for s, namely $s = -\ell, \ell+1$. The first solution yields a divergent solution, near $r = 0$ and so is discarded. Thus, $s = \ell + 1$ and we obtain $\eta(r) \sim r^{\ell+1}$ as $r \to 0$. This prompts us to consider the following yielding

$$\psi = \exp(-\sqrt{\lambda}r)\eta(r) = r^{\ell+1}\exp(-\sqrt{\lambda}r)\sum_{q=0}^{\infty} c_q r^q$$

6.3 Power series solution

From above we are looking for a solution for

$$\eta(r) = r^{\ell+1}\sum_{q=0}^{\infty} c_q r^q$$

Substituting η into the equation

$$\left[-\frac{d^2}{dr^2} + 2\sqrt{-\lambda}\frac{d}{dr} + \frac{\ell(\ell+1)}{r^2} - \frac{\kappa}{r} -\right]\eta = 0$$

we obtain

$$\sum_{q=0}^{\infty}[-(q+\ell+1)(q+\ell)c_q r^{q+\ell-1} + 2\sqrt{-\lambda}(q+\ell+1)c_q r^{q+\ell}$$
$$+\ell(\ell+1)c_q r^{q+\ell-1} - \kappa c_q r^{q+\ell}] = 0$$

By combining the terms that contain the same power of r,

$$\sum_{q=0}^{\infty}\left[-q(q+2\ell+1)c_q r^{q+\ell-1} + (2\sqrt{-\lambda}(q+\ell+1) - \kappa)c_q r^{q+\ell}\right] = 0$$

Then, shifting the summation index $q \to q-1$ [3] in the second term $r^{q+\ell}$ of the above sum, the equation becomes:

$$\sum_{q=0}^{\infty} \left[-q(q+2\ell+1)c_q + (2\sqrt{-\lambda}(q+\ell) - \kappa)c_{q-1} \right] r^{q+\ell-1} = 0$$

Therefore, we deduce that

$$-q(q+2\ell+1)c_q + (2\sqrt{-\lambda}(q+\ell) - \kappa)c_{q-1} = 0$$

and this is a recursion relation between the coefficients of the Taylor expansion of

$$\frac{\eta(r)}{r^{\ell+1}} = \sum_{q=0}^{\infty} c_q r^q$$

Considering the limit as $q \to \infty$,

$$q(q+2\ell+1)c_q = (2\sqrt{-\lambda}(q+\ell) - \kappa)c_{q-1}$$

$$\frac{c_q}{c_{q-1}} = \frac{2\sqrt{-\lambda}(q+\ell) - \kappa}{q(q+2\ell+1)}$$

$$\frac{c_q}{c_{q-1}} \sim \frac{2\sqrt{-\lambda}}{q}, \qquad c_q \sim \frac{(2\sqrt{-\lambda})^q}{q!}$$

The asymptotic behaviour for c_q yields a solution $\eta(r) \sim r^{\ell+1} exp(2\sqrt{-\lambda} r)$, which conversely, yielding a ψ that is non-normalisable. Hence, there must be $c_q \neq 0$ for some finite value of q (denoting them by $q = n > 0$), according to the equation

$$-q(q+2\ell+1)c_q + (2\sqrt{-\lambda}(q+\ell) - \kappa)c_{q-1} = 0$$

this can only happen if:

$$2\sqrt{-\lambda}(q+\ell) - \kappa = 0 \implies 2\sqrt{-\lambda} = \frac{\kappa}{n+\ell} \implies \lambda = -\frac{\kappa^2}{4(n+\ell)^2}$$

As a result, the expansion in $\eta(r) = r^{\ell+1} \sum_{q=0}^{\infty} c_q r^q$ only contains a finite number of terms, and it is therefore, a polynomial in r of finite order n. We can also see that as

$$H\psi = \lambda\psi \qquad \text{and} \qquad \lambda = -\frac{\kappa^2}{4(n+\ell)^2},$$

the spectrum is discrete, and

$$\sigma_{disc}(H_\ell) = -\frac{c}{(n+\ell)^2}, \forall n \in \mathbb{N}.$$

6.4 Laguerre polynomials

We have found in the previous section that the solution has to be of the form

$$\psi_{\ell,m} = r^{\ell+1} e^{-\frac{\kappa r}{2(\ell+1)}} f_m,$$

where f_m is a polynomial of degree $m = n, n-1, \ldots, 1, 0$ corresponding to the states of $\ell = 0, 1, \ldots, n-1$. We substitute this form of the solution into the modified equation

$$\left(-\frac{d^2}{dr^2} + 2\sqrt{-\lambda} + \frac{\ell(\ell+1)}{r^2} - \frac{\kappa}{r} \right) \eta = 0$$

where $\eta = r^{\ell+1} f$ and f is a polynomial of order n. Firstly, we compute the derivatives of η, we have $\eta' = r^{\ell+1}\left(\frac{\ell+1}{r} f + f'\right)$ and $\eta'' = r^{\ell+1}\left(\frac{\ell(\ell+1)}{r^2} + 2\frac{\ell+1}{r} f' + f''\right)$. Hence the differential equation from above becomes

$$\begin{aligned}
&- f'' - \frac{2(\ell+1)}{r} f + \frac{2(\ell+1)\sqrt{-\lambda}}{r} f + 2\sqrt{-\lambda} - \frac{\kappa}{r} f = 0 \\
&\iff r f'' + f'(2\ell + 2 - 2\sqrt{-\lambda} r) + f((\ell+1)2\sqrt{-\lambda} - \kappa) = 0
\end{aligned}$$

which is similar to the equation of the Laguerre polynomials

$$x L_n^\alpha(x)'' + (\alpha + 1 - x) L_n^{\alpha\prime} + n L_n^\alpha = 0.$$

We aim to find suitable substitutions for r such that our differential equation becomes of the form

$$x y'' + (1 + k - x) y' + j y = 0,$$

which gives us as the solution the Laguerre polynomial L_j^k.
Notice that $y_j^k(x) = e^{-x/2} x^{\frac{k+1}{2}} L_j^k(x)$ solves the equation

$$y_j^{k\prime\prime} + \left(-\frac{1}{4} + \frac{2j+k+1}{2x} - \frac{k^2-1}{4x^2} \right) y_j^k = 0. \tag{13}$$

We can easily compute the derivatives of y_j^k. For the simplicity of writing we denote $y_j^k = y$ and $L_j^k = v$. We have:

$$\begin{aligned}
&y' = e^{-x/2} x^{\frac{k+1}{2}} \left(-\frac{1}{2} v + \frac{k+1}{2x} v + v' \right) \iff e^{x/2} x^{-\frac{k+1}{2}} y' = -\frac{1}{2} v + \frac{k+1}{2x} v + v' \\
&e^{x/2} x^{-\frac{k+1}{2}} y'' = -\frac{1}{2} v' - \frac{k+1}{2x^2} v' + v'' + \frac{1}{4} v - \frac{k+1}{4x} v - \frac{1}{2} v' - \frac{k+1}{4x} + \frac{(k+1)^2}{4x^2} v + \frac{k+1}{2x} v.
\end{aligned}$$

If we plug this in the 13, we obtain:

$$-\frac{1}{2} v' - \frac{k+1}{2x^2} v' + v'' + \frac{1}{4} v - \frac{k+1}{4x} v - \frac{1}{2} v' - \frac{k+1}{4x} + \frac{(k+1)^2}{4x^2} v + \frac{k+1}{2x} v + \left(-\frac{1}{4} + \frac{2j+k+1}{2x} - \frac{k^2-1}{4x^2} \right) v = 0$$

which reduces itself after simplifications to

$$v'' - v' + \frac{k+1}{x}v' + \frac{j}{x}v = 0 \Longleftrightarrow xv'' + (k+1-x)v' + jv = 0.$$

Hence, if our solution is of the form $\phi = e^{-x/2}x^{\frac{k+1}{2}}v$, then $v = L_j^k$ is a Laguerre polynomial.

It remains to find suitable substitutions which can lead us to the exact form of the differential equation which is solved by the Laguerre polynomial. The method used is based on the idea found in the article [12]. Let us start again from the initial equation

$$-\frac{d^2\psi}{dr^2} + \frac{\ell(\ell+1)\psi}{r^2} - \frac{\kappa\psi}{r} - \lambda\psi = 0, \tag{14}$$

where $\lambda = -\frac{\kappa^2}{4(\ell+n)^2}$. We make the change of variable $x = r\frac{\kappa}{\ell+n}$. Let us denote $\frac{\kappa}{n+\ell} = \epsilon$. Hence the derivatives will be$dr = \frac{dx}{\epsilon}$ and $\frac{d^2\psi(r)}{dr^2} = \epsilon^2\frac{d^2\psi(x)}{dx^2}$. Therefore 14 becomes

$$\begin{aligned}
&\epsilon^2\frac{d^2y(x)}{dx^2} + \left(\frac{\kappa\epsilon}{x} - \frac{\epsilon^2}{4} + \epsilon^2\frac{\ell(\ell+1)}{x^2}\right)y(x) = 0 \\
&y'' + \left(-\frac{1}{4} + \frac{\kappa}{\epsilon x} - \frac{\ell(\ell+1)}{x^2}\right)y = 0.
\end{aligned}$$

Notice that that we can put the condition

$$\ell(\ell+1) = \frac{k^2-1}{4} \Longleftrightarrow k = 2\ell+1$$

and

$$\frac{\kappa}{\epsilon} = \frac{2j+k+1}{2}.$$

Hence, using this change of notations we obtain that our solution is of the form

$$y_j^k(x) = e^{-x/2}x^{(k+1)/2}L_j^k(x).$$

Another way of writing the coefficients is by using $k = 2\ell+1$, which implies that $\frac{2j+k+1}{2} = j+\ell+1 = n$. Hence the Laguerre polynomial can be written as follows $L_{n-\ell-1}^{2\ell+1}$, provided that $n-\ell-1$ is not a negative integer, in other words $\ell = 0, 1, \ldots, n-1$. Additionally $2\ell+1 > 0$ (by a condition of the Laguerre polynomial), which is true by the initial condition of $\ell > -\frac{1}{2}$.

6.5 Darboux Solution

We understand from the discussion regarding the Hamiltonian that Q and Q^* behave similarly to the Poschl-Teller potential, in that Q^*Q and QQ^* yield $H_\ell + M$ and $H_{\ell+1} + M$, respectively, where M denotes the constant $\frac{\kappa^2}{4(\ell+1)^2}$

This relationship allows us to approach the problem in a similar manner to the Poschl-Teller potential, in that we know that for each ℓ, H_ℓ has the same spectrum as $H_{\ell+n}$, where $n \in Z$, excluding the case in which $\lambda = 0$.

We begin by applying the same method as the Poschl-Teller case, finding the ground state for generic ℓ

$$Q_\ell \psi_1^{(\ell)} = 0 \implies (\frac{d}{dr} - \frac{\ell+1}{r} + \frac{\kappa}{2(\ell+1)})\psi_1^{(\ell)} = 0$$

And up to a multiplicative constant, it is quick to verify that $\psi_1^{(\ell)} = r^{\ell+1}e^{-\frac{\kappa}{2(\ell+1)}}$ satisfies this equation for arbitrary ℓ and κ.

We now seek to determine how applications of Q^* influences the solution

$$\psi_1^{(\ell)},\ \psi_2^{(\ell)}, \psi_3^{(\ell)} \ldots \xrightarrow{Q^*_{\ell+1}} \psi_2^{(\ell+1)},\ \psi_3^{(\ell+1)},\ \psi_4^{(\ell+1)},\ \ldots \xrightarrow{Q^*_{(\ell+2)}} \psi_3^{(\ell+2)},\ , \psi_4^{(\ell+2)},\ \psi_5^{(\ell+2)},$$

And in this process a method of recovering $\psi_n^{(\ell)}$ reveals itself, by applying operators Q_n^* strategically. Indeed, application of Q^*_{l+1} shifts the state from l to $l+1$, and also increases the excitement state of the particle. Explicitly:

$$\psi_n^{(\ell)} = Q_\ell^* \psi_{n-1}^{(l-1)} = Q_\ell^* Q_{\ell-1}^* ... Q_{\ell-n+2}^* \psi_1^{\ell-n+1}$$

And in general this yields:

$$\psi_n^{(\ell)} = \left(-\frac{d}{dr} - \frac{\ell+1}{r} + \frac{k}{2(\ell+1)}\right)\left(-\frac{d}{dr} - \frac{\ell}{r} + \frac{k}{2(\ell)}\right)\cdots\left(\frac{d}{dr} - \frac{\ell-n}{r} + \frac{k}{2(\ell-n)}\right) r^{\ell+1}e^{-\frac{\kappa r}{2(\ell+1)}}$$

As with the harmonic oscillator case, we can approach these solutions as being Laguerre Polynomials multiplied by the ground state inductively.

7 The ground state has no zero

Considering again the one-dimensional Schrödinger equation for some potential $V(x)$:

$$-\psi'' + V(x)\psi = \lambda\psi$$

An interesting result is that the ground state of the equation never assumes the value zero (i.e. the eigenfunction ψ_1 corresponding to the lowest eigenvalue λ_1 is non-zero). And we can prove this result by contradiction.
Since we are working with the L^2 space, we may scale a solution ψ with a constant and have $\int |\psi|^2 dx = 1$.
Then multiplying our equation by ψ on both sides :

$$(-\psi'' + V(x)\psi)\psi = (\lambda\psi)\psi$$

Taking λ to be the lowest eigenvalue λ_1 and integrating both sides with respect to x from $-\infty$ to ∞:

$$\int_{-\infty}^{\infty}(-\psi'' + V(x)\psi)\psi \; dx = \int_{-\infty}^{\infty}(\lambda_1\psi)\psi \; dx$$

$$\int_{-\infty}^{\infty}(-\psi\psi'' + V|\psi|^2)dx = \lambda_1 \int_{-\infty}^{\infty}|\psi|^2 dx$$

Integrating $\int_{-\infty}^{\infty}(\psi\psi'')dx$ by parts:

$$u = \psi, \; dv = \psi'' dx, \quad du = \psi' dx, \; v = \psi'$$

$$\int_{-\infty}^{\infty}(\psi\psi'')dx = [\psi\psi']_{-\infty}^{\infty} - \int_{-\infty}^{\infty}|\psi'|^2 dx$$

$$[-\psi\psi']_{-\infty}^{\infty} + \int_{-\infty}^{\infty}(|\psi'|^2 + V|\psi|^2)dx = \lambda_1 \cdot 1$$

Therefore, we obtain

$$\lambda_1 = \inf_{\psi,\psi' \in L^2(R)} \int_{-\infty}^{\infty}(|\psi'|^2 + V|\psi|^2)dx$$

as λ_1 is the smallest eigenvalue[6].
Then we define $\phi = \frac{\psi+|\psi|}{2}$, the equation of λ_1 above also holds true for ϕ[6].
As a result, ϕ must also be an eigenfunction with the eigenvalue λ_1, because the infimum of the spectrum λ_1 is achieved on the Sobolev class (H denotes the Hilbert space):

$$H^1(R) = \{\psi : \|\psi\|_{L^2} + \|\psi'\|_{L^2} < \infty\}$$

[6]
However, when ψ is negative, $\phi = \frac{\psi+|\psi|}{2} = \frac{\psi-\psi}{2} = 0$, and this is impossible for ϕ to be an eigenfunction with the same eigenvalue λ_1 as the ground state ψ because a part of ϕ is identically zero, and if a function f satisfies the Schrodinger equation and that both $f = 0$ and $f' = 0$ at a point, then f is the constant zero function.[2]
This is a contradiction, as ϕ is positive where $\psi > 0$, while it is also required that ψ is identially zero.
Thus, the ground state ψ of the one-dimensional Schrödinger equation cannot have a value zero.

References

[1]

[2] M Cohen. The energy spectrum of the excitations in liquid helium. 1956. https://thesis.library.caltech.edu/1007/1/Cohen$_{m1}$956.*pdf*.

[3] L Del Debbio. *Quantum Mechanics*. University of Edinburgh. https://www2.ph.ed.ac.uk/ ldeldebb/docs/QM/lect11.pdf.

[4] R.L. Jaffe. An algebraic approach to reflectionless potentials in one dimension. 2009.

[5] Priv-Doz Dr Martin Kohlmann. Schrodinger operators and their spectra. *Vorlesungsskript, Göttingen, Wintersemester*, 18, 2017.

[6] A. Laptev. Email sent to yuhao liu, June 2022.

[7] A. Laptev. Project meeting, June 2022.

[8] A. Laptev. Project meeting, June 2022.

[9] V. B. Matveev and M.A. Salle. *Darboux transformations and solitons*. Springer-Verlag, 1991.

[10] V. Moscovich. Norm, strong, and weak operator topologies on b (h). 2009.

[11] Jet Nestruev. *Cutoff and Other Special Smooth Functions on* $R^n.Springer, 2020.$

[12] University of Washington. Hydrogen atom, chapter 10. 2009. https://faculty.washington.edu/seattle/physics227/reading/reading-26-27.pdf.

[13] A Pankov. Introduction to spectral theory of schrödinger operators. 2006.

[14] P Rodriguez. *MATH50006 Measure and Integration Lecture notes*. Imperial College London, 2022.

[15] W. Rudin. *Functional Analysis*. International series in pure and applied mathematics. McGraw-Hill, 1991. https://books.google.co.uk/books?id=Sh_vAAAAMAAJ.

[16] J Sakurai and J Napolitano. Modern quantum mechanics. 2-nd edition. *Person New International edition*, pages 89–97, 2014.

[17] Charles Swartz. *An introduction to functional analysis*, volume 157. CRC Press, 1992.

[18] D Tong. *Quantum Mechanics*. University of Cambridge, 2021. https://www.damtp.cam.ac.uk/user/tong/qm/qm2.pdf.

[19] Hermann Weyl. Über gewöhnliche differentialgleichungen mit singularitäten und die zugehörigen entwicklungen willkürlicher funktionen. *Mathematische Annalen*, 68(2):220–269, 1910.