Solitary Waves and the KdV Equation

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Outline

In 1834, Victorian naval architect and engineer Scott Russell observed solitary waves that propagated with unchanging form and with amplitude-dependent speed [1].

Goal: To develop a model that accounts for such a phenomenon in shallow waves with uniform cross section and examine some of its analytical as well as numerical properties.

Background

Following Jog and Chandrashekhar (2002, p.353–355), water is taken to behave like an incompressible and irrotational Newtonian fluid. Additionally, making the notion of shallow waves precise, we introduce $\epsilon = a/h << 1$ and $\delta = (h/\lambda)^2 << 1$ as in figure 1 [2]. Thus, the governing equations are [3]:

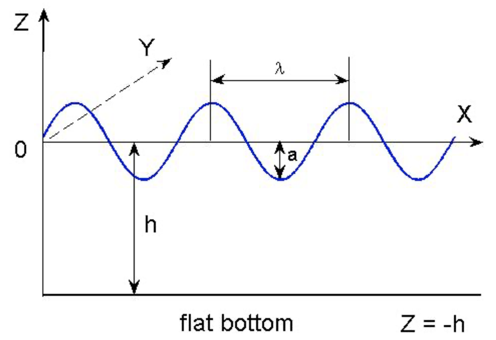

Figure 1: Periodic shallow wave moving in the X axis [1]

$$\begin{cases} \rho(\partial_t \mathbf{u} + (\mathbf{u}.\nabla)\mathbf{u}) = -\nabla \mathbf{P} + \rho \mathbf{g} \\ \nabla.\mathbf{u} = \mathbf{0} \\ \nabla \times \mathbf{u} = \mathbf{0} \end{cases} \tag{1}$$

for $-h \leq z \leq \eta(x,t)$, where η is the surface of the water satisfying $\eta \sim O(a)$ and the density ρ is uniform. The conditions in (1) imply that $\mathbf{u} = \nabla\phi$ for some $\phi = \phi(x,z,t)$.

The particles on the surface stay on the surface and the seabed is assumed impermeable [3]. Hence, the boundary conditions at the surface of the water and at the seabed are:

$$\begin{cases} \partial_z \phi = \partial_t \eta + \partial_x \phi \partial_x \eta \\ \partial_z \phi(x, z = -h, t) = 0 \end{cases} \tag{2}$$

Thus, the above governing equations become:

$$\begin{cases} \nabla(\partial_t \phi + \frac{1}{2}\|\nabla\phi\|^2 + P/\rho + gz) = \mathbf{0} \\ \nabla^2 \phi = 0 \end{cases} \tag{3}$$

The boundary conditions (2) and the governing equations (3) imply that ϕ when expanded about $z = -h$ has the form:

$$\phi = \sum_{m=0}^{\infty} \frac{(-1)^m (z+h)^{2m}}{(2m)!} \partial_x^{(2m)} f \tag{4}$$

To make use of the variables ϵ and δ, one rescales the variables involved into dimensionless ones (below they will be denoted the same as the previous variables) to make the scales apparent leading to the following system of PDE's to first order in ϵ and δ:

$$\begin{cases} \partial_t \omega - \delta/2 \partial_{xxt}\omega + \epsilon\omega\partial_x\omega + \partial_x \eta = 0 \\ \partial_t \eta + \partial_x \omega + \epsilon\omega\partial_x\eta + \epsilon\eta\partial_x\omega - \delta/6 \partial_{xxx}\omega = 0 \end{cases} \tag{5}$$

where $\omega = \partial_x f$. Using the ansatz to first order:

$$\omega = \eta + \epsilon F(x,t) + \delta G(x,t) \tag{6}$$

gives the KdV equation cast in dimenisonless parameters $x, t, \psi(\eta)$ [2]:

$$\partial_t \psi + \psi \partial_x \psi + \partial_{xxx} \psi = 0 \tag{7}$$

A simpler Model-Linearised KdV Equation

A simplification to equation (7) can be made by removing the non-linear advection term leading to:

$$\partial_t u + \partial_{xxx} u = 0 \tag{15}$$

$$\begin{cases} \text{u(x,0) = f(x), x} \in \mathbb{R} \\ \text{u} \to 0 \text{ as } |x| \to 0 \text{ for } t > 0 \end{cases} \tag{16}$$

Fokas and Ablowitz (2003, p.446) apply Fourier transforms of both sides of (15) gives:

$$\partial_t \hat{u}(k,t) - ik^3 \hat{u}(k,t) = 0 \tag{17}$$

with initial condition $\hat{u}(k,0) = \mathcal{F}\{u(x,0)\}$ which is a third order linear ODE in $\hat{u}(k,t) = \mathcal{F}\{u(x,t)\}$ and can readily be solved giving

$$\hat{u}(k,t) = \mathcal{F}\{u(x,0)\} \exp(ik^3 t) \tag{18}$$

Applying the inverse Fourier Transform to $\hat{u}(k,t)$ gives:

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(k,0) e^{i(k^3 t + kx)} dk \tag{19}$$

Upon making the substitution $\xi = (3t)^{\frac{1}{3}} k$ and setting $\eta = \frac{x}{(3t)^{\frac{1}{3}}} \sim O(1)$, $u(x,t)$ becomes:

$$u(x,t) = \frac{1}{2\pi (3t)^{\frac{1}{3}}} \int_{-\infty}^{\infty} \hat{u}\left(\frac{\xi}{(3t)^{\frac{1}{3}}}, 0\right) e^{i\left(\frac{\xi^3}{3} + \xi\eta\right)} d\xi \tag{20}$$

Taylor expanding $\hat{u}\left(\frac{\xi}{(3t)^{\frac{1}{3}}}, 0\right)$ about $\xi = 0$, one obtains as $t \to \infty$ [6]:

$$\begin{aligned} u(x,t) &= \frac{\hat{u}(0,0)}{2\pi(3t)^{\frac{1}{3}}} \int_{-\infty}^{\infty} e^{i\left(\frac{\xi^3}{3} + \xi\eta\right)} d\xi \\ &+ \frac{\hat{u}'(0,0)}{2\pi(3t)^{\frac{2}{3}}} \int_{-\infty}^{\infty} \xi e^{i\left(\frac{\xi^3}{3} + \xi\eta\right)} d\xi + \dots \\ &= \frac{\hat{u}(0,0)}{(3t)^{\frac{1}{3}}} Ai(\eta) + O\left(\frac{1}{(3t)^{\frac{2}{3}}}\right) \end{aligned} \tag{21}$$

where Ai is the Airy function. The above analysis suggests that for $t >> 1$, the dominant contribution is made by the first term in (21) and is of order $\frac{1}{(3t)^{\frac{1}{3}}}$ in a region of width $O((3t)^{\frac{1}{3}})$. The combination of both oscillatory and exponential behaviour of the Airy function seem to indicate that the solution will asymptotically disperse into smaller waves, as evidenced in figure 4 [7].

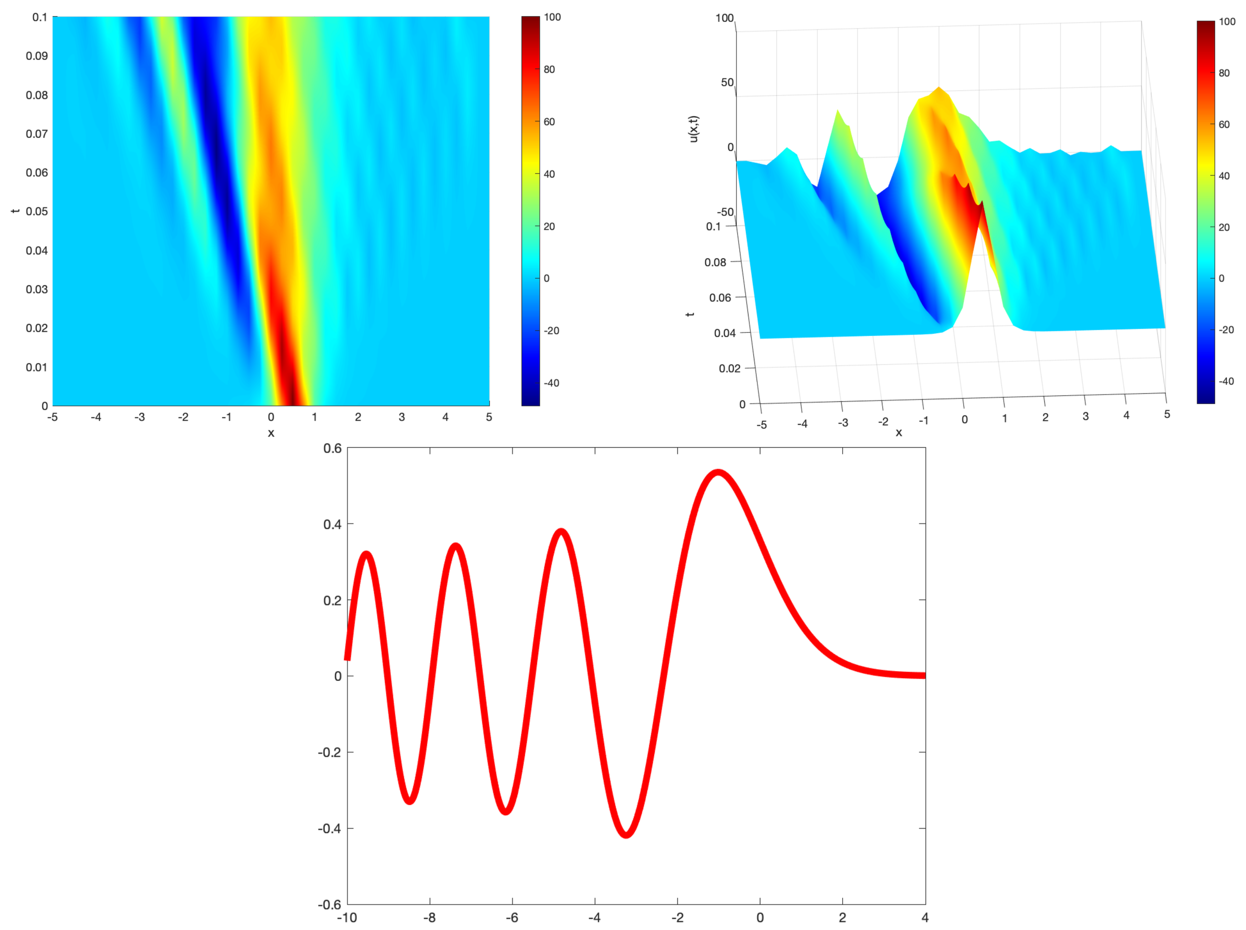

Figure 4: Solitary wave initial condition ($\sim \text{sech}^2$) for linear KdV (top view left, side view right) and plot of the Airy function (bottom) (MATLAB).

Travelling wave Solutions

Travelling wave solutions have the form $u(x,t) = \sigma(\xi)$ where $\xi = x - ct$ and c is interpreted as the speed of the wave. Requiring that $u(x,t)$ satisfy (7), one obtains: $-c\sigma + \sigma\sigma' + \sigma''' = 0$. Integrating twice yields:

$$(\sigma')^2 = c(\sigma)^2 - \frac{1}{3}(\sigma)^3 + K_1\sigma + K_2 \tag{8}$$

where K_1 and K_2 are constants of integration. For a solitary wave (a soliton) that vanishes asymptotically, one imposes that $\sigma, \sigma', \sigma'' \to 0$ as $|\xi| \to \infty$ means that $K_1, K_2 = 0$. Hence, using separation of variables, σ satisfies

$$u(x,t) = \sigma(\xi) = 3c\,\text{sech}^2\left[\frac{\sqrt{c}}{2}(x - ct - x_0)\right] \tag{9}$$

where x_0 is a constant of integration. The speed c of the wave depends linearly on the amplitude of the wave A as $c = \frac{A}{3}$. Additionally, one can pick K_1 and K_2 such that in (8) $G(\sigma) = c(\sigma)^2 - \frac{1}{3}(\sigma)^3 + K_1\sigma + K_2 = -\frac{1}{3}(\sigma - g_1)(\sigma - g_2)(\sigma - g_3)$ with $g_3 < g_2 < g_1$. Using the same methods as the soliton case, one obtains for $u(x,t)$ [4]:

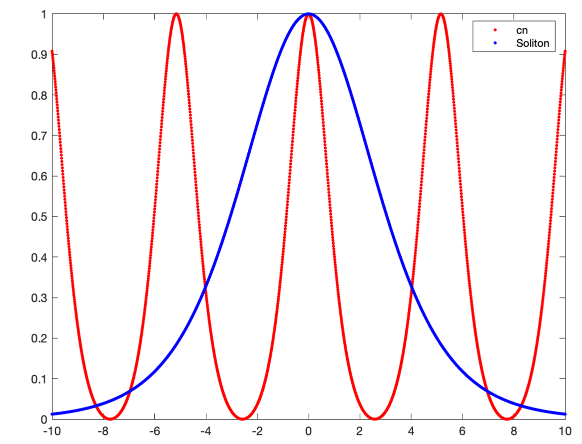

Figure 2: Cnoidal wave (red) and soliton (blue) (MATLAB)

$$u(x,t) = \sigma(\xi) = g_2 - (g_2 - g_3)cn^2\left[\frac{(x - ct - \xi_0)}{\sqrt{12}}\sqrt{g_1 - g_3}; m\right] \tag{10}$$

where $cn(x;m) = \cos(E^{-1}(\psi;m))$, a Jacobi Elliptic function that is periodic and yields the desired periodic solution. Note the amplitude and speed are independent, unlike the soliton solution.

Numerical Results-Finite Difference Approximation

In general, it is not easy to find analytical solutions to the KdV equation and a numerical approach is warranted. Following Zabusky and Kruskal [5], the (bounded) domain is partitioned into a mesh of points $(i\Delta x, j\Delta t) \equiv (x_i, t_j)$ - with uniform spacing and the approximation to the solution at these points is given by $U_i^j \approx u(i\Delta x, j\Delta t)$.

In discretising space, the following approximations are used:

$$u(x_i, t_j) \approx \frac{1}{3}[u(x_{i+1}, t_j) + u(x_i, t_j) + u(x_{i-1}, t_j)] \tag{11}$$

$$\begin{aligned} \partial_{xxx} u|_{(x_i,t_j)} \approx \frac{1}{2\Delta x^3} &[u(x_{i+2}, t_j) \\ &- 2u(x_{i+1}, t_j) + 2u(x_{i-1}, t_j) - u(x_{i-2}, t_j)] \end{aligned} \tag{12}$$

where (11) is the three-point average approximation for $u(x,t)$ and (12) is the centred finite difference approximation to $\partial_{xxx} u(x,t)$. In discretising the derivative with respect to time at interior points, one has:

$$\partial_t u|_{(x_i,t_j)} \approx \frac{U_i^{j+1} - U_i^{j-1}}{2\Delta t} \tag{13}$$

Substituting the above in (7) gives the scheme:

$$\begin{aligned} U_i^{j+1} = U_i^{j-1} &- 1/3\frac{\Delta t}{\Delta x}(U_{i+1}^j + U_i^j + U_{i-1}^j)(U_{i+1}^j - U_{i-1}^j) \\ &- \frac{\Delta t}{\Delta x^3}(U_{i+2}^j - 2U_{i+1}^j + 2U_{i-1}^j - U_i^j - 2) \end{aligned} \tag{14}$$

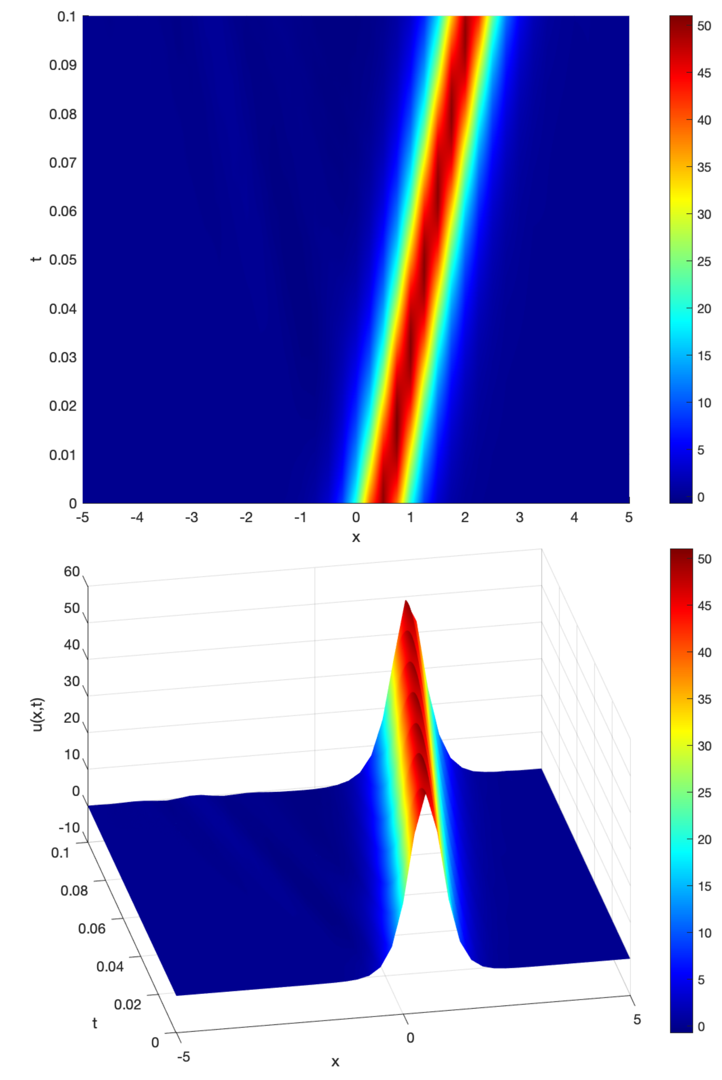

Figure 3: Soliton initial condition propagated forward in time from t = 0 to t = 0.015 with $\Delta t = 0.000001$ and $\Delta x = 0.2$. Top view(left), side view (right) using the Zabusky-Kruskal explicit scheme (MATLAB)

Outlook

In conclusion, upon developing a model for such phenomena, the attempt at linearisation does not seem to capture fully the 'solitary waves' of Scott Russell as dispersion and decay seemed to be dominant features. Thus, it becomes clear that the non linear advection term $u \times \partial_x u$ is necessary for a complete description of solitons. Solitons were however discovered by examining travelling wave solutions to the non-linear KdV equation and the desirable properties were also verified by numerical simulation. Extensions to the above analysis could include:

1. A more general boundary condition at $z = -h$ to include a variable sea bed given by $b(x,t)$ [1]
2. A modified KdV equation by adding a diffusive $-u^p \partial_{xx} u(x,t)$ with $p \in \mathbb{Z}_{>2}$ term [5]
3. A generalisation of the KdV equation to two dimensions [5].

References

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