Chapter 4

* Define Brownian Snake (path valued process), study its basic properties
  - f.d.d.
  - Strong Markov.
* Exhibit alternative construction of SBM using Brownian Snake.
* Explore sample path properties using Brownian Snake representation.

1. Brownian Snake

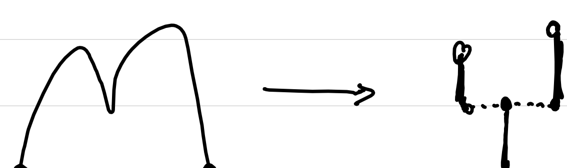
where branching mechanism Ψ "encodes" probability & # of offspring or death.

(Galton-Watson)

Motivation: SBM arose as scaling limit of continuous-time branching processes. Can one obtain a more "dynamical" representation of SBM where at time $t>0$, "SBM_t corresponds to "descendants" at time $t>0$ of some branching mechanism?

Q: How to define a Brownian motion on a 'genealogical' structure associated to some lifetime process $f \in C(\mathbb{R}_+;\mathbb{R}_+)$? (Say $f(0)=0$)

A: Obtain tree from f by identifying values s, s' if $f(s), f(s')$ are 'visible' from one another, i.e. $f(s)=f(s')=\inf_{s\le t\le s'} f(t)$



and 'scan tree from left to right' obtaining a path-valued process $(W_s)_{s\ge 0}$ with life-time $W_s : [0, f(s)] \to \mathbb{R}^d$, $d\in\mathbb{N}$. Then, at time $t>0$, denoting $\hat{W}_s$, the terminal point of W_s, $W_s(f(s))\ (=: \hat{W}_s)$, the 'descendants' of $W_0(0) := x \in \mathbb{R}^d$ are located at
$\{\hat{W}_s : s\ge 0, f(s)=t\}$.

We now construct the Brownian snake more formally. We begin with some definitions. Let

$$\mathcal{W} := \bigcup_{t \geq 0} \mathbb{D}([0,t], \mathbb{R}^d)$$

where for an interval $I \subseteq \mathbb{R}_+$, E metric space, we write $\mathbb{D}(I, E)$ for the space of cadlag paths on I taking values in E. For $w \in \mathcal{W}$, define its <u>lifetime</u> $\zeta_w \in \mathbb{R}_+$ by

$$\zeta_w := \inf\{t \geq 0 : w(\cdot) = w(\cdot \wedge t)\}$$

<u>Rmk</u>: 1) $\mathcal{W}$ inherits a topology from a complete metric ρ (in terms of ζ_w and Skorokhod topology on $\mathbb{D}([0,\infty); \mathbb{R}^d)$) turning $(\mathcal{W}, \rho)$ into a Polish space.

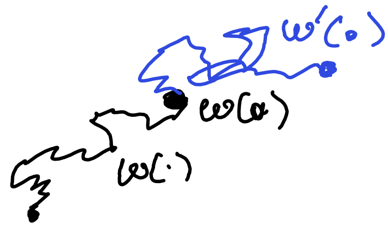
2) Can identify points $x \in \mathbb{R}^d$ with path $w \equiv x \in \mathcal{W}$ ($\zeta_w = 0$) and denote $\mathcal{W}_x := \{w \in \mathcal{W} : w(0) = x\}$.

Now, let $w \in \mathcal{W}$ and $a \in [0, \zeta_w]$, $b \geq a$. Define the prob. measure $R_{a,b}(w, dw')$ on $\mathcal{W}$ by:

recall Π_w law of B

1) $\zeta_{w'} = b$, $R_{a,b}(w, dw')$ a.s.,
2) $w'(t) = w(t)$, $\forall t \in [0,a]$, $R_{a,b}(w, dw')$ a.s.
3) Law $(w'(a+t), 0 \leq t \leq b-a)$ = Law $(B_t, 0 \leq t \leq b-a)$ under $\Pi_{w(a)}$

So



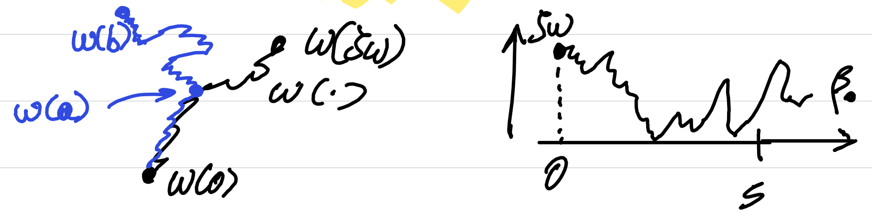
"Take $w(\cdot)$, chop off before life time at time $= a$ and run BM until time $= b \geq a$".

Let $(\beta_s, s \geq 0)$ be a reflected linear BM (i.e. $(\beta_s, s \geq 0) \overset{d}{=} (|B_s|, s \geq 0)$, $(B_s, s \geq 0)$ a standard BM) started at $x \in \mathbb{R}_{\geq 0}$. For every $s > 0$, denote by $\gamma_s^x(da, db)$ the joint distr. of the pair $(\inf_{0 \leq r \leq s} \beta_r, \beta_s)$. The reflection principle gives explicit form:

$$\gamma_s^x(da, db) = 2(x+b-2a) \exp\left(-\frac{(x+b-2a)^2}{2s}\right) \mathbb{1}(0 < a < b \wedge x)\, da\, db + 2(2\pi s)^{-1/2} \exp\left(-\frac{(x+b)^2}{2s}\right) \mathbb{1}(0 < b)\, \delta_0(da)\, db.$$

Definition: The Brownian snake is the Markov process in $\mathcal{W}$, denoted by $(W_s, s \ge 0)$ whose transition kernels $(Q_s, s \ge 0)$ are given by

$$Q_s(w, dw') = \iint \gamma_s^{\zeta_w}(da\,db)\, R_{a,b}(w, dw').$$



"Life time $\zeta_s := \zeta_{W_s} \overset{d}{=}$ reflected BM and when ζ_s increases, W_s evolves like BM, when ζ_s decreases, erase W_s."

Rmk: Collection of kernels forms semi-group, so existence of $(W_s, s \ge 0)$ as Markov process on $\mathcal{W}^{\mathbb{R}_+}$ is guaranteed by Kolmogorov's extension theorem.

There is an alternative construction of $W := (W_s, s \ge 0)$, which will be useful for the remainder of this talk and involves defining conditional distributions of W given the 'genealogical' structure (or lifetime process $(\zeta_s, s \ge 0)$).

Fix $w_0 \in \mathcal{W}$ and set $\zeta_0 = \zeta_{w_0}$. Will construct W on canonical space $C(\mathbb{R}_+; \mathbb{R}_+) \times \mathcal{W}^{\mathbb{R}_+}$. Let P_{ζ_0} denote law of refl. BM on $C(\mathbb{R}_+; \mathbb{R}_+)$ started at ζ_0. Then, for $f \in C(\mathbb{R}_+; \mathbb{R}_+)$, let $\Theta^f_{w_0}(dw)$ be the law on $\mathcal{W}^{\mathbb{R}_+}$ of the time-inhomogeneous Markov process in $\mathcal{W}$ started at w_0, whose transition kernel between times s, s' is

$$R_{m(s,s'),\, f(s')}(w, dw'),$$

where $m(s,s') = \inf_{s \le r \le s'} f(r)$. Existence of $\Theta^f_{w_0}(dw)$ follows from Kolmogorov ext. thm and meas. $f \mapsto \Theta^f_{w_0}(A)$, A Borel is also meas. (by monotone class thm). So makes sense to consider

$$\mathbb{P}_{w_0}(df\,dw) = P_{\zeta_0}(df)\, \Theta^f_{w_0}(dw).$$

Observe under $\mathbb{P}_{w_0}$, $\zeta_s(f,w) = f(s)$ a.s. So lifetime process is distributed like refl. BM.

Rmk: Kernels $(Q_s, s > 0)$ are symmetric wrt (invariant) measure

$$M(dw) = \int_0^\infty da\, R_{0,a}(x, dw).$$

We will henceforth be interested in excursion measures of the Brownian snake.

<u>Definition:</u> (Excursion measure) For $x\in\mathbb{R}^d$, $d\in\mathbb{N}$, let $\mathbb{N}_x$ denote the σ-finite measure

$$\mathbb{N}_x(d\zeta\, dw) = n(d\zeta)\,\Theta_x^{\zeta}(dw),$$

where $n(d\zeta)$ is the Itô excursion measure as in lecture 3.

<u>Lemma:</u> (i) $\forall \varepsilon>0,\ \alpha>0$, (uniformly in $w_0\in\mathcal{W}$)

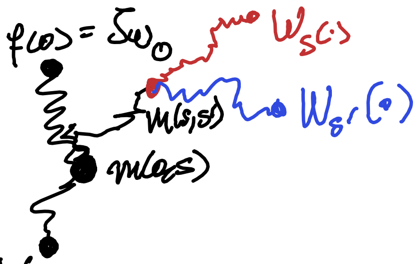
$$\lim_{r\downarrow 0}\left(\sup_{s\geq\alpha}\mathbb{P}_{w_0}\left(d(W_s,W_{s+r})>\varepsilon\right)\right)=0.$$

(ii) Let $f\in C_c(\mathbb{R}_+;\mathbb{R}_+)$, $f(0)=0$. Then $\forall\varepsilon>0$,

$$\lim_{r\downarrow 0}\left(\sup_{s\geq 0}\Theta_x^{f}\left(d(W_s,W_{s+r})>\varepsilon\right)\right)=0,$$

where the convergence is uniform in $x\in\mathbb{R}^d$ and rate only depends on the modulus of cont. of f.

<u>Proof (Sketch)</u> (i): Fix $0<s<s'$. Given $f\in C(\mathbb{R}_+;\mathbb{R}_+)$ s.t. $m(0,s)\leq m(s,s')$, then under Θ_x^f, $W_s, W_{s'}$ agree up to $m(s,s')$ and then evolve independently as Brownian motions.



Then can estimate $\forall\varepsilon,\eta>0$

$$\mathbb{P}_{w_0}\left(\sup_{t\geq 0}\left|W_s(t\wedge\zeta_s)-W_{s'}(t\wedge\zeta_{s'})\right|>2\varepsilon\right)$$

$$\leq P_{\zeta_{w_0}}\left(m(s,s')<m(0,s)\right)+P_{\zeta_{w_0}}\left(\zeta_s-m(s,s')>\eta\right)+P_{\zeta_{w_0}}\left(\zeta_{s'}-m(s,s')>\eta\right)$$

$$+2E_{\zeta_{w_0}}\left(\mathbb{1}(m(0,s)\leq m(s,s'))\cdot\Pi_{w_0(m(0,s))}\left(\Pi_{\xi_{m(s,s')-m(0,s)}}\left(\sup_{0\leq r\leq\eta}|\xi_0-\xi_r|>\varepsilon\right)\right)\right)$$

where all terms go to zero as $\eta\searrow 0$, $s'-s\searrow 0$ and $s\geq\alpha>0$. $\square$

For (ii), argue similarly.

<u>Rmk:</u> 1) Part (i) of this lemma implies that $(W_s; s\geq 0)$ is cont. in prob. away from $s=0$. Can check that when w_0 has jump at ζ_{w_0}, then W_s is <u>not</u> cont. in prob. at $s=0$.

2) This lemma allows us to construct a jointly measurable modification W' of W (i.e. $\mathbb{P}_{w_0}(W'_s\neq W_s)=0$ $\forall s\geq 0, w_0\in\mathcal{W}$) and $\Theta_x^f(W'_s\neq W_s)=0$ $\forall s\geq 0$, $x\in E$, $n(df)$ a.e. $\rightsquigarrow$ diagonal argument, see Le Gall p. 57).

Recall the notation $\sigma := \sigma(f)$ under $\mathbb{N}_x$ for the 'terminal' value of the excursion f.

2. Finite dimensional marginals of the Brownian snake.

Goal: Characterise finite-dimensional marginals of Brownian Snake in terms of marked trees from previous lecture.

Let $\mathcal{T}_p$ denote the set of marked trees with $p \geq 1$ branches. Fix $\theta \in \mathcal{T}_p$, $x \in \mathbb{R}^d$ and let

$$\Pi_x^\theta,$$

denote a measure (to be constructed) on $(\mathcal{W}_x)^p$.

$p=1$: $\theta = (\{\emptyset\}, h)$ for some $h \geq 0$, then

set $\Pi_x^\theta := \Pi_x^h$, i.e. the law of $(\xi_t, 0 \leq t \leq h)$.

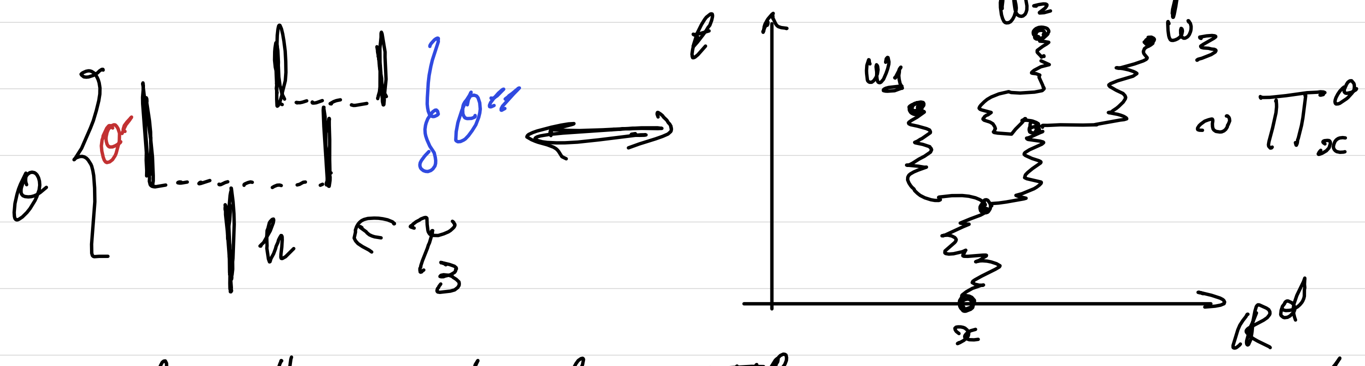
$p \geq 2$: θ can be expressed in a unique way as $\theta = \theta' \circledast_h \theta''$ $\to$ concatenation from previous lecture

where $\theta' \in \mathcal{T}_j$, $\theta'' \in \mathcal{T}_{p-j}$, $j \in \{1, \dots, p-1\}$. Then, define

$$\Pi_x^\theta := \int \Pi_x^\theta(dw_1, \dots, dw_p) F(w_{1:p})$$

$$= \Pi_x\Big(\iint \Pi_{\xi_h}^{\theta'}(dw'_{1:j}) \, \Pi_{\xi_h}^{\theta''}(dw''_{1:p-j}) \cdot F(\xi[0,h] \odot w'_{1:j}, \xi[0,h] \odot w''_{1:p-j})\Big)$$

$\boxed{\odot \text{ means concatenation of paths}}$



Informally, sample from Π_x^θ by running independent copies of ξ on branches of tree θ.

<u>Proposition</u>: (Brownian snake f.d.d.)

(i) let $f \in C(\mathbb{R}_+;\mathbb{R}_+)$, $f(0)=0$ and let $0 \le t_1 \le t_2 \le \cdots \le t_p$.
Then the law under Θ_x^f of $(\omega(t_1), \ldots, \omega(t_p))$ is $\Pi_x^{\theta(f,t_1,t_2,\ldots,t_p)}$. ($(W_t)_{t\ge 0} \sim$ Brownian snake).

(ii) For any $F \in \underbrace{B_+}_{\text{bdd Borel}}(\mathcal{W}_x^p)$,

$$N_x\left(\int_{\{0\le t_1\le t_2\le\cdots\le t_p\le\sigma\}} dt_{1:p}\, F(W_{t_{1:p}})\right) = 2^{p-1}\int \underbrace{\Lambda_p(d\theta)}_{\text{uniform measure on } T_p}\Pi_x^\theta(F).$$

<u>Proof</u>: (i) Follows from the def^n of Θ_x^f and construction of trees $\theta(f,t_{1:p})$. Can proceed as usual with induction on p and use Markov property.

(ii)

$$N_x\left(\int_{0\le t_1\le t_2\le\cdots\le t_n\le\sigma} dt_{1:p}\, F(W_{t_{1:p}})\right)$$

$$= \int n(df)\int_{\{\cdot\}} dt_{1:p}\, \Theta_x^f(F(W_{t_{1:p}}))$$

$$= \int n(df)\int_{\cdot} dt_{1:p}\, \Pi^{\theta(f,t_{1:p})}(F) \quad \text{part (i)}$$

$$= 2^{p-1}\int \Lambda_p(d\theta)\Pi_x^\theta(F) \quad \text{Thm III.4 in LeGall.}$$

<u>Rmk</u>: With $p=1, p=2$, can compute $\forall g \in B_+(\mathbb{R}^d)$

$$N_x\left(\int_0^\sigma ds\, g(\underbrace{\hat{W}_s}_{\text{terminal value}})\right) = \Pi_x\left(\int_0^\infty dt\, g(\xi_t)\right),$$

and $N_x\left(\left(\int_0^\sigma ds\, g(\hat{W}_s)\right)^2\right) = 4\Pi_x\left(\int_0^\infty dt\left(\Pi_{\xi_t}\left(\int_0^\infty dr\, g(\xi_r)\right)\right)^2\right).$

Formulas "look like" moment formulas for super processes obtained in Lecture 2 (See Chapter 2 LeGall), i.e.

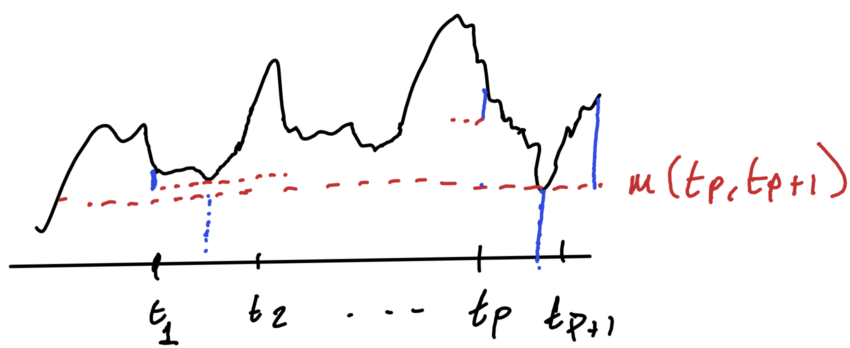
Quadratic branching, $\psi(u) = \beta u^2$,
$\forall g \in B_+(\mathbb{R}^d)$

$$\mathbb{E}_\mu\langle Z_t, f\rangle = \langle \mu, \Pi_\cdot f(\xi_t)\rangle$$

and

$$\mathbb{E}_\mu\langle Z_t, f\rangle^2 = \langle \mu, \Pi_\cdot f(\xi_t)\rangle^2 + 2\beta\int_0^t \langle \mu, \Pi_\cdot\left(\Pi_{\xi_s} f(\tilde{\xi}_{t-s})\right)^2\rangle ds.$$

Sketch of proof of (i):



Points t_p, t_{p+1} stay together until level $m(t_p, t_{p+1})$, when they branch and get separated and so conclude induction step using Markov ppty of $(W_s, s \geq 0)$ and definition of $\Pi_x^{\theta(f; t_{1:p+1})}$.

We now compute "Laplace-type" functionals involving "traces" of the Brownian snake, useful in obtaining (the representation of super BM in terms of the Brownian snake

Proposition 3: Let $g \in \mathcal{B}_{b+}(\mathbb{R}^d)$ s.t. $g(t,y) = 0$ for $t \geq A > 0$. Then the function

$$u_t(x) = \mathbb{N}_x\left(1 - \exp - \int_0^\sigma g(t+\zeta_s, \hat{W}_s)\right) \quad (*)$$

solves the integral equation

$$u_t(x) + 2\,\Pi_{t,x}\left(\int_t^\infty dr\, (u_r(\xi_r))^2\right) = \Pi_{t,x}\left(\int_t^\infty dr\, g(r,\xi_r)\right)$$

(recall ξ starts from x at time t under $\Pi_{t,x}$).

Proof: Idea is to write exponential in $(*)$ as a power series

$$u_t^\lambda(x) := \mathbb{N}_x\left(1 - \exp\left(-\lambda \int_0^\sigma g(t+\zeta_s, \hat{W}_s)\right)\right) = \sum_{p=1}^\infty (-1)^{p+1} \lambda^p\, T^p g(t,x),$$

where for every $p \geq 1$, we set

$$T^p g(t,x) = \frac{1}{p!}\, \mathbb{N}_x\left(\left(\int_0^\sigma ds\, g(t+\zeta_s, \hat{W}_s)\right)^p\right).$$

Then, using characterisation of fin. dim. marginals of W_s in terms of marked trees (sampled "uniformly") to obtain a recurrence relation for $T^p g$ in terms of $T^j g$ and $T^{p-j} g$, $1 \leq j \leq p-1$.

Then, obtaining a uniform bound for $T^p g(t,x)\, (\leq K \mathbb{1}_{[0,A]}(t)$ for some $K < \infty)$ can use Fubini etc. to obtain functional eq^n for u^λ for λ suff. small and obtain it for $\lambda = 1$ by analytic continuation

We prove the aforementioned recurrence a little more formally.

By $p=1$ case from Proposition 2 (ii) (in LeGall's book) have

$$T^1 g(t,x) = \Pi_x \left(\int_0^\infty dr\, g(t+r, \xi_r) \right).$$

For $p \geq 2$, using Prop. 2 (ii) again, have

$$T^p g(t,x) = \mathbb{N}_x \left(\int_{0 \leq s_1 \leq s_2 \leq \dots \leq s_p \leq \sigma} ds_1 \dots ds_p \prod_{i=1}^p g(t+\zeta_{s_i}, \hat{W}_{s_i}) \right)$$

$$= 2^{p-1} \int \Lambda_p(d\theta) \int \Pi_x^\theta (dw_{1:p}) \prod_{i=1}^p g(t+\zeta_{w_i}, \hat{w}_i)$$

$$= 2^{p-1} \sum_{j=1}^{p-1} \int_0^\infty dh \iint \Lambda_j(d\theta') \Lambda_{p-j}(d\theta'')$$

$$\times \Pi_x \Bigg(\Big(\int \Pi_{\xi_h}^{\theta'} (dw'_{1:j}) \prod_{i=1}^j g(t+h+\zeta_{w_i'}, \hat{w}_i') \Big)$$

$$\times \Big(\int \Pi_{\xi_h}^{\theta''} (dw''_{1:j}) \prod_{i=1}^{p-j} g(t+h+\zeta_{w_i''}, \hat{w}_i'') \Big) \Bigg).$$

Last equality follows from identity (by construction)

$$\Lambda_p = \sum_{j=1}^{p-1} \int_0^\infty dh\, \Lambda_j *_h \Lambda_{p-j}$$

This is because one can "sample" $\theta \in \mathbb{T}_p$ as $\theta = \theta' *_h \theta''$, where $\theta' \in \mathbb{T}_J$, $\theta'' \in \mathbb{T}_{p-J}$, $J \in \{1, \dots, p-1\}$ unif. and $h \sim$ Leb. s.t. $h \perp\!\!\!\perp J$ and cond on $J=j$. $\theta' \sim \Lambda_j$, $\theta'' \sim \Lambda_{p-j}$.

This gives recursive formula:

$$T^p g(t,x) = 2 \sum_{j=1}^{p-1} \Pi_x \left(\int_0^\infty dh\, T^j g(t+h, \xi_h)\, T^{p-j} g(t+h, \xi_h) \right)$$

$\square$

We now construct the super Brownian motion with quadratic branching mechanism from the Brownian snake.

<u>Theorem 4:</u> Let $\mu \in \mathcal{M}_b(\mathbb{R}^d)$ (bdd meas. in $\mathbb{R}^d$) and let

$$\mathcal{N} := \sum_{i \in I} \delta_{(x_i, f_i, w_i)} \text{ be a P.P.P.}$$

with intensity measure $\nu = \mu(dx) \mathbb{N}_x(df\, dw)$
can take law $(x_i, f_i, w_i) \sim \frac{\nu(\cdot \cap E_i)}{\nu(E_i)}$ $\hookrightarrow$ Brownian snake excursion measure.

Write $W_s^i = W_s(f_i, w_i)$, $\zeta_s^i = \zeta_s(f_i, w_i)$ and $\sigma_i = \sigma(f_i)$ $\forall i \in I$, $s \geq 0$. Then, there exists a $(\xi, 2u^2)$ superprocess
$\hookrightarrow$ BM

$(Z_t, t \geq 0)$ with $Z_0 = \mu$ such that for every $h \in \mathcal{B}_{b+}(\mathbb{R}_+)$ and $g \in \mathcal{B}_{b+}(\mathbb{R}^d)$,

$$\int_0^\infty h(t) \langle Z_t, g \rangle \, dt = \sum_{i \in I} \int_0^{\sigma_i} h(\zeta_s^i)\, g(\hat{W}_s^i)\, ds. \quad (*)$$

More precisely, by occupation density formula for local times, one can <u>define</u> for $t > 0$, the random measures $\forall g \in \mathcal{B}_b(\mathbb{R}^+)$

$$\langle Z_t, g \rangle = \sum_{i \in I} \int_0^{\sigma_i} d\ell_s^t(\zeta^i)\, g(\hat{W}_s^i), \text{ where}$$

$\ell_s^t(\zeta^i)$ denotes the local time at level t and at time s of $(\zeta_r^i, r \geq 0)$.

<u>Rmks:</u>

Excursion

(i) local times can be defined in terms of the usual representation: (can take jointly continuous version)

$$\ell_s^t(\zeta^i) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^s dr\, \mathbb{1}_{[t, t+\varepsilon)}(\zeta_r^i)$$

and $(\ell_s^t(\zeta^i), s \geq 0)$ is a continuous increasing function, for every $i \in I$, a.s. (Also, map $(t, a) \mapsto \ell_t^a(\zeta)$ is cont.)
(See Le Gall 1993, Brownian excursions, trees and measure-valued branching processes).

(ii) $(\lambda Z_t, t \geq 0)$ is a $(\xi, 2\lambda u^2)$-super process.

Proof: Let $\mathcal{L}$ denote the random measure on $\mathbb{R}_+\times\mathbb{R}^d$ defined by

$$\int \mathcal{L}(dt\,dy)h(t)g(y) = \sum_{i\in I}\int_0^{\sigma_i} h(\zeta_s^i)g(\hat{W}_s^i)ds,$$

for $h\in\mathcal{B}_{b+}(\mathbb{R}_+)$ and $g\in\mathcal{B}_{b+}(\mathbb{R}^d)$. Suppose that h is compactly supported. Then, by Campbell's thm for P.P.Ps and Prop. 3 (in LeGall's book — technical result computing first moment of Laplace-type functional of "trace-like" quantities of Brownian snake, in terms of sol^{n} to functional equation), one obtains

$$E\exp-\int\mathcal{L}(dt\,dy)h(t)g(y)$$

Campbell $\hookrightarrow$ $$= \exp\left(-\int\mu(dx)\,\mathbb{N}_x\left(1-\exp-\int_0^\sigma h(\zeta_s)g(\hat{W}_s)\right)\right)$$

$$= \exp(-\langle\mu,u_0\rangle)$$

where $(u_t(x), t\geq 0, x\in\mathbb{R}^d)$ is the unique non-negative solution of

$$u_t(x) + 2\Pi_{t,x}\left(\int_t^\infty dr\,(u_r(\xi_r))^2\right) = \Pi_{t,x}\left(\int_t^\infty dr\,h(r)g(\xi_r)\right).$$

By Cor II.9 (comparing terms), have that random meas.

$$\mathcal{L} \stackrel{d}{=} dt\,Z'_t(dy),$$

where Z' is a $(\xi, 2u^2)$-superprocess with $Z'_0=\mu$.

Since Z' is continuous in prob. (Prop II.8 in LeGall's book), we easily obtain that, for every $t\geq 0$, $\hookrightarrow$ actually can construct cadlag modification (Fitzsimmons, 1988).

$$Z'_t = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon} Z'_r\,dr, \text{ in prob. } (*)$$

It follows that $\forall\, t>0$, the limit

$$Z_t(dy) := \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathcal{L}(dr\,dy) \quad \text{exists in prob.}$$

Clearly, $Z\stackrel{d}{=}Z'$ (as processes) and so is also a $(\xi, 2u^2)$-superprocess started at μ.

Thus, have for all $t>0$, $g\in C_{b+}(\mathbb{R}^d)$,

$$\langle Z_t, g\rangle = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\mathcal{L}(dr\,dy)\mathbb{1}_{[t,t+\varepsilon]}(r)g(y)$$

$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\sum_{i\in I}\int_0^{\sigma_i} ds\,\mathbb{1}_{[t,t+\varepsilon]}(\zeta_s^i)g(\hat{W}_s^i).$$

(in prob.)

Note that $\exists$ finite number of non-zero terms in the sum over $i \in I$ since for $t>0$, $\mathbb{N}_x(\sup \zeta_s \geq t) = n(\sup e(t) > 0) < \infty$).

Now, we <u>claim:</u>

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\sigma ds\, \mathbb{1}_{[t,t+\varepsilon]}(\zeta_s)\, g(\hat{W}_s) = \int_0^\sigma d\ell_s^t(\zeta)\, g(\hat{W}_s)$$

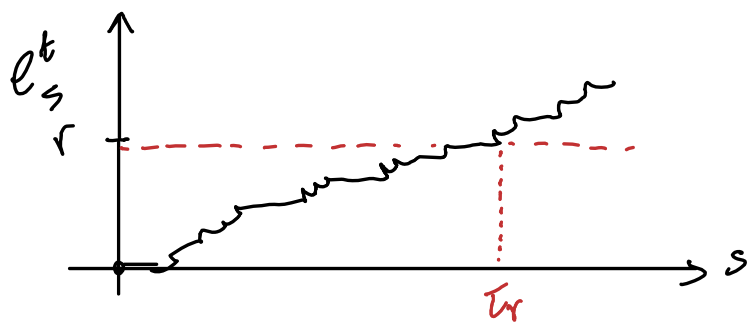
in $\mathbb{N}_x$-measure, for every $x \in \mathbb{R}^d$.

To see this, note the following equalities

$$\int_0^\sigma d\ell_s^t(\zeta)\, g(\hat{W}_s) = \int_0^\infty dr\, \mathbb{1}_{\{\tau_r < \infty\}}\, g(\hat{W}_{\tau_r}),$$

$$1/\varepsilon \int_0^\sigma ds\, \mathbb{1}_{[t,t+\varepsilon]}(\zeta_s)\, g(\hat{W}_s) = \int_0^\infty dr\, \mathbb{1}(\tau_r^\varepsilon < \infty)\, g(\hat{W}_{\tau_r^\varepsilon}),$$

where $\tau_r = \inf\{s : \ell_s^t > r\}$, $\tau_r^\varepsilon = \inf\{s : \frac{1}{\varepsilon}\int_0^s du\, \mathbb{1}_{[t,t+\varepsilon]}(\zeta_u) > r\}$.



To see this, we establish the following deterministic result. Let $\ell(\cdot)$ be a continuous non-dec. function on $\mathbb{R}_+$ and let $\tau_r := \inf\{s \geq 0 : \ell(s) > r\}$. Suppose also $\mathrm{supp}(d\ell) \subset\subset \mathbb{R}_+$. Then, for any $\sigma \geq \mathrm{diam}(\mathrm{supp}(d\ell))$, and $g \in B_{b+}(\mathbb{R}_+)$,

$$\int_0^\sigma g(s)\, d\ell_s = \int_0^\infty dr\, \mathbb{1}(\tau_r < \infty)\, g(\tau_r).$$

Proof: Observe that on continuity pts. of $(\tau_r, r\geq 0)$, $s = \tau_{\ell_s}$. Since ℓ is monotone $\Rightarrow$ set of discontinuities of τ is at most countable. So, can decompose

$$\int_0^\sigma g(s)\,d\ell_s = \sum_{i\in I} \int_{a_i}^{b_i} g(s)\,d\ell_s, \text{ where on } (a_i, b_i),$$

I countable

ℓ is strictly increasing. Hence,

$$\int_0^\sigma g(s)\,d\ell_s = \sum_{i\in I} \int_{a_i}^{b_i} g(\tau_{\ell_s})\,d\ell_s = \int_0^\infty g(\tau_{\ell_s})\mathbb{1}(\tau_{\ell_s}<\infty)\,d\ell_s$$

$\cdot \mathbb{1}(\tau_{\ell_s}<\infty)$

(since on supp$(d\ell)$, $s = \tau_{\ell_s}$). Now, by the change of variables formula (under map $s \mapsto \ell_s$, $d\ell \mapsto (\ell_s)^* d\ell = \text{Leb}$), have

$$\int_0^\sigma g(s)\,d\ell_s = \int_0^\infty g(\tau_{\ell_s})\cdot \mathbb{1}(\tau_{\ell_s}<\infty)\,d\ell_s$$

$$= \int_0^\infty g(\tau_r)\cdot \mathbb{1}(\tau_r<\infty)(\ell_*)\,d\ell$$

$$= \int_0^\infty g(\tau_r)\cdot \mathbb{1}(\tau_r<\infty)\,dr \; .$$

Now, $\tau_r^\varepsilon \to \tau_r$ as $\varepsilon \to 0$, N_x a.e. on $\{\tau_r<\infty\}$. Holds a.e. for r cont. points (since $\tau_{r^-} \leq \liminf \tau_r^\varepsilon \leq \limsup \tau_r^\varepsilon \leq \tau_r$).

$$\int_0^\infty |g(\hat{W}_{\tau_r^\varepsilon})\mathbb{1}(\tau_r^\varepsilon<\infty) - g(\hat{W}_{\tau_r})\mathbb{1}(\tau_r<\infty)|\,dr$$

$$\leq \underbrace{\int_0^{\sup_s \ell_s} \cdots}_{\text{goes to } 0 \text{ as } \varepsilon\downarrow 0 \text{ by above remark}} + \|g\|_\infty \int_{\sup_s \ell_s}^{\sup_{0\leq s\leq\sigma} \frac{1}{\varepsilon}\int_0^s \mathbb{1}_{(t,t+\varepsilon)}(\zeta_s)\,ds \,\vee\, \sup_s \ell_s} \mathbb{1}(\tau_r^\varepsilon<\infty)\,dr$$

$$\leq \|g\|_\infty \cdot \left(\sup_{0\leq s\leq\sigma} \frac{1}{\varepsilon}\int_0^s \mathbb{1}_{(t,t+\varepsilon)}(\zeta_s)\,ds - \sup_{0\leq s\leq\sigma} \ell_s\right)$$

$$\underset{\text{occupation time density}}{\leq} \|g\|_\infty \cdot \left(\sup_{0\leq s\leq\sigma} \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \ell_s^a\,da - \sup_{0\leq s\leq\sigma} \ell_s^t\right) \to 0 \text{ as } \varepsilon\to 0$$

Now, can use bounded convergence to establish $(\mu \in M_b(\mathbb{R}^d))$

$$\lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^d} \mu(dx) \int_{\sup f\geq t} n(df) \int_{\mathcal{W}} \Theta_x^f(dw) \int_0^\infty |g(\hat{W}_{\tau_r^\varepsilon})\mathbb{1}(\tau_r^\varepsilon<\infty) - g(\hat{W}_{\tau_r})\mathbb{1}(\tau_r<\infty)| = 0 .$$

To see this, section with $\{\sup_s \zeta_s \geq t\}$, $\{\ell_\sigma^t \leq M\}$ use bdd conv. and then take $M \uparrow \infty$. Note on $\{\sup_s \zeta_s \geq t\}$, N_x is a finite measure.

This gives representation in terms of local times. Now, $(*)$ follows from the occupation time density formula for Brownian local times.

To see $(*)$, note that by Section 2 (LeGall) have that total mass $\langle Z_t, 1\rangle$ is a.s. bdd on $0 \le t \le T$ for any fixed $T>0$. (Can always choose modification s.t. above holds.) Now, for $g \in B_{bt}(\mathbb{R}^d)$, have for any $\delta > 0$ $(T > t)$

$$\limsup_{\varepsilon \downarrow 0} \mathbb{P}\left(\left| \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \langle Z_r, g\rangle - \langle Z_t, g\rangle \, dr \right| > \delta \right)$$

$$\le \limsup_{\varepsilon \to 0} \mathbb{P}\left(\sup_{0 \le r \le T} |\langle Z_r, 1\rangle| \ge M \right)$$

$$+ \limsup_{\varepsilon \to 0} \mathbb{P}\left(\frac{1}{\varepsilon} \int_t^{t+\varepsilon} |\langle Z_r, g\rangle - \langle Z_t, g\rangle| \cdot \mathbb{1}\left(\sup_{0 \le r \le T} |\langle Z_r, 1\rangle| \le M\right) dr > \delta \right)$$

(Markov + Fubini)

$$\le \mathbb{P}\left(\sup_{0 \le r \le T} |\langle Z_r, 1\rangle| \ge M \right)$$

$$+ \limsup_{\varepsilon \downarrow 0} \frac{1}{\delta} \sup_{t \le r \le t+\varepsilon} \mathbb{E}\left[|\langle Z_r, g\rangle - \langle Z_t, g\rangle| \cdot \mathbb{1}\left(\sup_{0 \le r \le T} |\langle Z_r, 1\rangle| \le M\right) \right]$$

$$\le \mathbb{P}\left(\sup_{0 \le r \le T} |\langle Z_r, 1\rangle| \ge M \right)$$

$$+ \frac{2M}{\delta}\eta + \limsup_{\varepsilon \to 0} \frac{2M}{\delta} \sup_{t \le r \le t+\varepsilon} \mathbb{P}\left(|\langle Z_r, g\rangle - \langle Z_t, g\rangle| > \eta \right)$$

$$= \left(\frac{2M+1}{\delta}\right)\eta + \mathbb{P}\left(\sup_{0 \le r \le T} |\langle Z_r, 1\rangle| \ge M \right) \longrightarrow 0 \text{ as}$$

$M \to \infty$ and $\eta \downarrow 0$

Sample path properties of Brownian snake

Recall we take $\xi = \text{BM}$ on $\mathbb{R}^d$, $d \geq 1$. Then, have that there exist constants $C, p > 2, \varepsilon > 0$ s.t.

$$\Pi_x\Big(\sup_{0 \leq r \leq t} |x - \xi_r|^p \Big) \leq C t^{2+\varepsilon}$$

(in particular, $\forall p > 4$, take $\varepsilon = p/2 - 2$).

We now show that the Brownian snake admits a continuous modification.

Proposition 5: The process $(W_s, s \geq 0)$ has a continuous modification under $\mathbb{N}_x$ or under $\mathbb{P}_x$ for every $x \in E, w \in \mathcal{W}$. (Here $w \in \bigcup_{t \geq 0} \mathcal{C}([0,t]; \mathbb{R}) \subseteq \mathcal{W}$).

Proof: Recall paths of (reflected) linear BM are a.s $1/2 - \eta$ Hölder cont for every $\eta > 0$. So, fix $f \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}_+)$ s.t. $\forall T > 0$ and $\eta \in (0, 1/2)$, there exists a constant $C_{\eta, T}$ with

$$|f(s) - f(s')| \leq C_{\eta,T} \cdot |s - s'|^{1/2 - \eta} \quad \forall s, s' \in [0, T].$$

Now, suffices to show that $\forall$ such f, Θ^f_w descends onto a measure on $\mathcal{C}(\mathbb{R}_+; \mathcal{W})$, for any w s.t. $\zeta_w = f(0)$.

Suppose first $f(0) = 0$, and so $w \equiv x \in \mathbb{R}^d$. By construction, have that the joint law $(W_s, W_{s'})$ for $s < s'$ under Θ^f_x is

$$\Pi^{f(s)}_x(dw)\, R_{m(s,s'), f(s')}(w, dw').$$

Then, $\forall s, s' \in [0, T]$, $s \leq s'$,

$$\Theta^f_x\big(d(W_s, W_{s'})^p\big) \leq$$

$$c_p\Big(|f(s) - f(s')|^p + 2\Pi_x\Big(\Pi_{\xi_{m(s,s')}}\Big(\sup_{0 \leq t \leq (f(s) \vee f(s')) - m(s,s')} |\xi_0 - \xi_t|^p \Big)\Big)\Big)$$

(recall $d(w, w') = |\zeta_w - \zeta_{w'}| + \sup_{t \geq 0} |w(t \wedge \zeta_w) - w'(t \wedge \zeta_{w'})|$).

$$\leq c_p\Big(|f(s) - f(s')|^p + 2C\,|(f(s) \vee f(s')) - m(s,s')|^{2+\varepsilon} \Big)$$

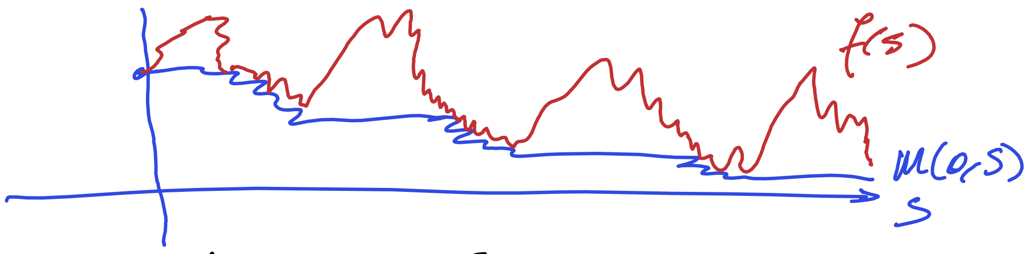
$$\leq c_p\Big(C^p_{\eta,T} |s - s'|^{p(\frac{1}{2} - \eta)} + 2C\, C^{2+\varepsilon}_{\eta,T} |s - s'|^{(2+\varepsilon)(\frac{1}{2} - \eta)} \Big),$$

now, with $\eta > 0$ small enough s.t. $p(1/2 - \eta) > 1$ and $(2+\varepsilon)(1/2 - \eta) > 1$, and $\varepsilon = p/2 - 2$, we have

$$\Theta^f_x\big(d(W_s, W_{s'})^p\big) \leq C_{\eta, p}(f)\, |s - s'|^{\frac{p}{2}(\frac{1}{2} - \eta)}.$$

Hence, by Kolmogorov's continuity theorem, have $(\forall T\geq 0)$ that $(W_s, 0\leq s\leq T)$ has a cont. mod. under $\mathbb{P}^f_x$ (which is Hölder cont. with exponent θ, $\forall\, 0<\theta<\frac{p(\frac{1}{2}-\varepsilon)-1}{2p}$ for all $p>4$ which gives $\theta = 1/4-\gamma$ for any $\gamma\in(0,1/4)$.)

Now, for $f(0)>0$, same argument gives that W_s has a cont. mod. on all $[a,b]$, where $f(s) > m(a,s)$ for $s\in(a,b)$.



On "excursion intervals" (a,b) s.t. $f(s)>m(a,s)$, $m(a,s)$ is constant and so $\forall s<s'\in(a,b)$ have $m(s,s')\geq m(0,s)$.

We thus estimate

$$d(W_s,W_{s'})^p \overset{?}{\leq} C_p\Big(|f(s)-f(s')|^p + \Pi_{\omega_0(m(0,s))}\Big(\Pi_{\xi}\max_{\substack{0\leq r\leq \eta \\ \geq m(s,s')-m(0,s)}}|\xi_0-\xi_r|^p\Big)\Big)$$

where $\eta = (f(s)\vee f(s')) - m(s,s')$

$$\overset{?}{\leq} C_p(|f(s)-f(s')|^p) + C_p|f(s)\vee f(s') - m(s,s')|^{2+\varepsilon}$$

$\vdots$

Now define $W'_s := \begin{cases} \omega(\cdot\wedge f(s)), & f(s)=m(0,s). \\ W_s, & \text{otherwise} \end{cases}$

Easy to check $W'_\cdot$ is a modification of $W_\cdot$. Now, can safely assume $\mathbb{P}^f_{w}$ a.s. $\rightarrow W_s$ $\omega(\cdot\wedge f(s))\ \forall s$, s.t. $f(s)=m(0,s)$. Hence, for $s\leq s'$, with

$$\alpha := \inf\{r\geq s : f(r)=m(0,r)\}$$
$$\beta := \sup\{r\leq s' : f(r)=m(0,r)\},$$

we estimate:

$$d(W_s, W_{s'}) \leq d(W_\alpha, W_s) + d(W_\alpha, W_\beta) + d(W_\beta, s')$$

which gives that $(W_s, s\geq 0)$ has a continuous modification. (Upon observing that $\forall\,\omega\in\mathcal{W}$, $\lim_{\varepsilon\to 0} d_0(\omega(\cdot\wedge t), \omega(\cdot\wedge(t+\varepsilon))) = 0$).

Rmk: This construction gives that $\mathbb{P}_w$-a.s., or $\mathbb{N}_x$-a.e. $\omega\in\bigcup_{t\geq 0} C([0,t],\mathbb{R}^d)$, $\forall s<s'$, $W_s(t)=W_{s'}(t)$ $\forall t\leq m(s,s')$. (For fixed $s<s'$, this is clear by construction and by continuity, holds simultaneously for all $s<s'$. This is called the Snake property).

We now briefly remark that spatial (Hölder cont.) of sample paths of ξ gives regularity of Brownian snake in the form of a continuous modification.

For $\xi \overset{d}{=} BM_x$, this is immediate from previous construction involving local times and the fact that ($t \mapsto d\ell^t_S(s)$ is cont. from $\mathbb{R}_+$ into $\mathcal{M}_b(\mathbb{R}_+)$, N_x a.e.) $s \mapsto \hat{W}_s$ is continuous (prop. 5).

From now on, we will work only with such continuous modifications of $(W_s, s \geq 0)$!

We now state the Strong Markov Property of W, very useful in applications. First, some notation. Denote by $\mathcal{F}_s$ the σ-algebra generated by $(W_r, 0 \leq r \leq s)$ and as usual take

$$\mathcal{F}_{s+} := \bigcap_{r>s} \mathcal{F}_r.$$

<u>Theorem 6</u>: The process $(W_s, \mathbb{P}_w)$ is strong Markov wrt the filtration $(\mathcal{F}_{s+})_{s\geq 0}$.

<u>Proof</u>: Let T be a stopping time wrt $(\mathcal{F}_{s+})_{s\geq 0}$ s.t. $T \leq K$ for some deterministic $K<\infty$. Let F be bdd and $\mathcal{F}_{T+}$-meas. and Ψ a bdd measurable fn on $\mathcal{W}$. Enough to prove:

$$\mathbb{E}_w[F\Psi(W_{T+s})] = \mathbb{E}[F\,\mathbb{E}_{W_T}[\Psi(W_s)]].$$

Can assume Ψ is cont. Then have

$$\mathbb{E}_w[F\Psi(W_{T+s})] = \lim_{n\to\infty} \sum_{k=0}^{\infty} \mathbb{E}\left[\mathbb{1}\left(\tfrac{k}{n} \leq T < \tfrac{k+1}{n}\right) F\,\Psi(W_{\frac{k+1}{n}+s})\right]$$

Markov ppty and on $\{\frac{k}{n} \leq T < \frac{k+1}{n}\}$, F is $\mathcal{F}_{\frac{k+1}{n}}$-meas.

$$= \lim_{n\to\infty} \sum_{k=0}^{\infty} \mathbb{E}\left[\mathbb{1}\left(\tfrac{k}{n} \leq T < \tfrac{k+1}{n}\right) F\,\mathbb{E}_{W_{\frac{k+1}{n}}}[\Psi(W_s)]\right]$$

Now, have bound (on $\{k/n \leq T < k+1/n\}$)

$$\left|\mathbb{E}_{W_{k+1/n}}(\Psi(W_s)) - \mathbb{E}_{W_T}(\Psi(W_s))\right| \leq \sup_{\substack{t\leq K\\ t\leq r\leq t+1/n}} \left|\mathbb{E}_{W_r}(\Psi(W_s)) - \mathbb{E}_{W_t}(\Psi(W_s))\right|$$

Observe that by construction, can compute

$$\mathbb{E}_{W_r}(\Psi(W_s)) = \int \gamma_s^{\zeta_r}(da\,db) \int R_{a,b}(W_r, dw')\Psi(w')$$

and similarly for $\mathbb{E}_{W_t}(\Psi(W_s))$. Set

$$c(\varepsilon) = \sup_{t\leq K,\, t\leq r\leq t+\varepsilon} |\zeta_r - \zeta_t|$$

and note that $c(\varepsilon) \to 0$, $\mathbb{P}_w$-a.s. Then, observe that if $t \leq K$ and $t \leq r \leq t+\varepsilon$, the paths coincide at least until $(\zeta_t - c(\varepsilon))_+ \leq m(t, t+\varepsilon)$. Thus,

$$R_{a,b}(W_r, dw') = R_{a,b}(W_t, dw')$$

for all $a \leq (\zeta_t - c(\varepsilon))_+$, and $b \geq a$. Then, by a computation, (using explicit form of $\gamma_s^{\zeta_r}(da, db)$),

$$\lim_{\varepsilon\downarrow 0}\left(\sup_{t\leq K,\, t\leq r\leq t+\varepsilon} \left|\mathbb{E}_{W_r}(\Psi(W_s)) - \mathbb{E}_{W_t}(\Psi(W_s))\right|\right) = 0, \quad \mathbb{P}_w\text{-a.s.}$$

□

<u>Rmk</u>: Only need ξ to be Markov (to be able to construct kernels $R_{a,b}(w, dw')$), <u>not</u> strong Markov.

Rmk:

So now, W is a cont. strong Markov process.
Moreover, $\forall x \in \mathbb{R}^d$, (i.e. x is regular for W)
$$\mathbb{P}_x(T_{\{x\}}=0)=1, \text{ where}$$
$T_{\{x\}} := \inf\{s>0, W_s = x\}$ and follows directly from analogous ppty of reflected linear Brownian motion.

Thus, can consider excursion measure of W away from x, which coincides with $\mathbb{N}_x$.

Then, have the following form of the Strong Markov Property under the excursion measure $\mathbb{N}_x$. Let S be a stopping time wrt $(\mathcal{F}_{s+})$ s.t. $S>0$ $\mathbb{N}_x$ a.e., let G be a non-negative $\mathcal{F}_{S+}$-measurable r.v. and H be a non-negative meas. f^{n} on $C(\mathbb{R}_+; \mathcal{W})$.
Then,

$$\mathbb{N}_x(G\,H(W_{S+s}, s\geq 0)) = \mathbb{N}_x(G\,\mathbb{E}_{W_S}(H(W_{s\wedge T_{\{x\}}}, s\geq 0))).$$

We now use the construction of SBM using the Brownian snake to obtain some sample path properties.

Recall the random measures (under $\mathbb{N}_x$)

$$\langle Z_t, g\rangle = \int_0^{\sigma} d\ell_s^t(\zeta)\, g(\hat{W}_s).$$

for $g \in \mathcal{B}_{b+}(\mathbb{R}^d)$.

Let $\operatorname{supp} Z_t$ denote the topological support of Z_t and define the range $\mathcal{R}$ by

$$\mathcal{R} := \overline{\bigcup_{t \geq 0} \operatorname{supp} Z_t}.$$

Then, have the following result.

<u>Theorem 7:</u> the following hold $\mathbb{N}_x$-a.e. $\forall x \in \mathbb{R}^d$.

(i) The process $(Z_t, t \geq 0)$ has continuous sample paths.

(ii) $\forall t > 0$, $\operatorname{supp} Z_t \subset\subset \mathbb{R}^d$. If ξ is Brownian motion in $\mathbb{R}^d$, $\dim(\operatorname{supp} Z_t) = 2 \wedge d$ a.e. on $\{Z_t \neq 0\}$.

(iii) the set $\mathcal{R}$ is a connected compact subset of $\mathbb{R}^d$. If ξ is a BM in $\mathbb{R}^d$,

$$\dim(\mathcal{R}) = 4 \wedge d.$$

<u>Proof:</u> (i) By joint continuity of Brownian local times, the map $t \mapsto d\ell_s^t(\zeta)$ from $\mathbb{R}_+$ into $\mathcal{M}_f(\mathbb{R}_+)$ is continuous $\mathbb{N}_x$-a.e. By Prop. 5, $s \mapsto \hat{W}_s$ is continuous (having a continuous modification), whence the result follows.

(ii) $\forall t > 0$, $\operatorname{supp} Z_t \subset \{\hat{W}_s, 0 \leq s \leq \sigma\}$ which is compact (by Prop 5). Spse ξ is a BM in $\mathbb{R}^d$. By def^n, have

$$\operatorname{supp} Z_t \subset \{\hat{W}_s : 0 \leq s \leq \sigma, \zeta_s = t\}.$$

<u>Fact:</u> $\dim\{s \in [0,\sigma], \zeta_s = t\} \leq 1/2$ (level sets for BM have dim 1/2).

<u>Fact:</u> $\Pi_x\left(\sup_{0 \leq r \leq t} |\xi_r - x|^p\right) \leq C t^{2+\varepsilon}$ $\forall p > 4$, $\varepsilon = p/2 - 2$

so prop 5 gives

$$\Theta_x^f\left(d(W_s, W_{s'})^p\right) \leq C_p(f) \cdot |s - s'|^{\frac{p}{2}(\frac{1}{2} - \varepsilon)}$$

for $s, s' \in [0, \sigma(f)]$, $n(df)$ a.e. By Kolmogorov's continuity lemma, have $s \to W_s$ is Hölder cont. with exponent $1/4 - \gamma$ for all $\gamma > 0$. This Hölder reg. transfers to $s \to \hat{W}_s$ which gives upper bound

$$\dim\{\hat{W}_s : s \in [0,\sigma], \zeta_s = t\} \leq 4 \dim\{s \in [0,\sigma], \zeta_s = t\} \leq 2.$$