Imperial College London

COURSEWORK 1

IMPERIAL COLLEGE LONDON

DEPARTMENT OF MATHEMATICS

MATH70054
Introduction To SDEs And Diffusion Processes

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Following the lecture notes and [1], let X_t be a n-dimensional Itô process. That is,

$$dX_t = \boldsymbol{b}(t, X_t)dt + \boldsymbol{\sigma}(t, X_t)dW_t, \quad X_0 = x,$$

where $\boldsymbol{b}(\cdot,\cdot) : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\boldsymbol{\sigma}(\cdot,\cdot) : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, measurable vector and matrix valued functions respectively. In lectures, we have seen that the multi-dimensional Itô formula for the transformation $V(t,x) \in C^{1,2}([0,T], \mathbb{R}^n)$ is

$$dV(t, X_t) = \frac{\partial V}{\partial t}dt + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} dX_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 V}{\partial x_i \partial x_j} dX_i \cdot dX_j \tag{1}$$

where $dX_i \cdot dX_j$ is computed using the convention $dW_i \cdot dW_j = \delta_{ij}dt, dt \cdot dW_i = 0, dt \cdot dt = 0$, where $W_i, i = 1,\ldots,m$ are the components of an m-dimensional Brownian motion.

Problems

Question 1

Part (i)

Let X_t solve the Itô SDE

$$dX_t = (m - X_t)dt + \sigma dW_t, \quad X_0 = x \tag{2}$$

Find a closed form for X_t.

Part (ii)

CSuppose x is deterministic. Compute the mean and variance of X_t.

Part (iii)

Compute the second moment of X_t for more general initial conditions X_0 independent of the Brownian motion $\mathcal{W}_t$ driving 11.

Part (iv)

Find the law of X_t and compute asymptotically the weak limit of the law of X_t as $t \to \infty$.

Question 2

Part (a)

ai)

Find the generator of the Fokker-Planck equation corresponding to the linear SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x \tag{3}$$

where $b(X_t) = \mu X_t$ and $\sigma(X_t) = \sigma X_t$.

aii)

Consider for $n \geq 1, n \in \mathbb{N}$, $M_t^n = (X_t)^n$, where X_t is a solution to (12). Compute the $n-$th moment $\mathbb{E}[X_t^n]$.

aiii)

Suppose that for all $t > 0$, $X_t > 0$ a.s. where the process X_t solves SDE (12). By considering the SDE satisfied by the process consider $g(t, X_t) = \log(X_t) \in C^2$, find a closed form expression for X_t.

aiv)

Consider the Stratonovich interpretation of (12), namely

$$dX_t = \mu X_t dt + \sigma X_t \circ dW_t. \tag{4}$$

Obtain the equivalent Itô SDE corresponding to 13 .

Part (b)

Consider the SDE:

$$dX_t = f(t)X_t dt + h(t)X_t dW_t, \quad X_0 = x$$

for f, h continuous. By considering the SDE that the process $V(X_t, t) = \log(X_t)$ find a closed form expression for X_t.

Part(c)

Suppose X_t solves the SDE

$$dX_t = b(X_t)dt + X_t dW_t$$

Find the transformed SDE that $\log(X_t)$ satisfies.

Part (d)

Suppose now that X_t is a solution to

$$dX_t = \left(c + \frac{1}{2}\sum_{j=1}^{n} \alpha_j\right) X_t dt + X_t \sum_{j=1}^{n} \alpha_j dW_j(t)$$

By considering the function $V(x,t) = \log(x)$, solve the SDE.

Question 3

Part (a)

Find a solution to the SDE

$$dX_t = dt + 2\sqrt{X_t}dW_t, X_0 = 0 \quad \text{a.s.}$$

Part (b)

Suppose that X_t is a solution to the SDE:

$$dX_t = \left(\frac{1}{4} - X_t\right)dt + \sqrt{X_t}dW_t, \quad X_0 = x.$$

Find a closed form expression for X_t.

Question 4

Part (a)

Let Y_1, Y_2 be:

$$Y_1(t) = cos(W_t), \quad Y_2(t) = sin(W_t),$$

where W_t is a standard one-dimensional Brownian motion. Find a system of SDEs satisfied by (Y_1, Y_2).

Part (b)

Compute the generator for the SDE obtained in part (b).

Part (c)

Let X_t solve the SDE

$$\begin{cases} dX_1(t) = -X_2(t)dW_t - \frac{1}{2}X_1(t)dt, & X_1(0) = 1 \\ dX_2(t) = X_1(t)dW_t - \frac{1}{2}X_2(t)dt, & X_2(0) = 0. \end{cases}$$

Find a closed form expression for (X_1, X_2).

Question 5

Part (i)

We consider the Stratonovich SDE

$$dX_t = f(X_t) \circ dW_t, \quad X_0 = x \tag{5}$$

where f is positive and differentiable. From lectures, this SDE is equivalent to the following Itô SDE

$$dX_t = \frac{1}{2} f(X_t) f'(X_t) dt + f(X_t) dW_t$$

Now, using

$$h(x) = \int_0^x \frac{1}{f(z)} dz$$

solve the SDE.

Part (ii)

Solve the SDE

$$dX_t = -\frac{1}{2} X_t dt + \sqrt{1 - X_t^2} dW_t, \quad X_0 = x \in [-1, 1],$$

Question 6

In this question, we consider the following system of SDEs:

$$\begin{cases} dq_t = p_t dt, \quad q_0 = q, \\ dp_t = -q_t dt + z_t dt, \quad p_0 = p, \\ dz_t = -z_t dt - p_t dt + \sqrt{2} dW_t, \quad z_0 \sim \mathcal{N}(0, 1) \end{cases} \tag{6}$$

part (i)

Find the generator of the above SDE (18).

Part (ii)

By considering a suitable transformation, show that q_t, p_t satisfy the following SDE:

$$\begin{cases} \frac{dq_t}{dt} = p_t \\ \frac{dp_t}{dt} = -q_t - \int_0^t p_s e^{-(t-s)} ds + F(t) \end{cases} \tag{7}$$

Define the process $F(t)$ by

$$F(t) = z_0 e^{-t} + \sqrt{2} \int_0^t e^{-(t-s)} dW_s \tag{8}$$

Investigate the properties of $F(t)$. (Compute its mean, auto-correlation function and law).

Part (iii)

Is (q_t, p_t) a Markov process? What about (q_t, p_t, z_t)?

Question 7

Part (i)

Suppose X_t satisfies the SDE

$$dX_t = f(X_t)dt + g(X_t)dW_t \tag{9}$$

and let

$$Z(x) = \frac{f(x)}{g(x)} - \frac{1}{2}\frac{dg}{dx}(x).$$

By considering the function $V(t,x) = e^{\theta B(x)}$ for some θ to be determined, show one can pick θ in such a way so that the SDE satisfied by $V(t, X_t) = e^{\theta B(X_t)}$ is linear in V.

Part (ii)

Consider the SDE

$$dX_t = (\lambda X_t - X_t^2)dt + X_t dW_t \tag{10}$$

By considering the process $Y_t = V(t,x) = e^{\theta B(x)}$ for some θ to be determined, using the previous part find a linear SDE solved by Y_t.

Solutions

Question 1

Part (i)

Let X_t solve the Itô SDE

$$dX_t = (m - X_t)dt + \sigma dW_t, \quad X_0 = x \tag{11}$$

By considering the integrating factor e^t, we apply Itô's lemma to $Y_t = e^t X_t$ to obtain the following SDE for $Y_t = g(t, X_t)$ with $g(t,x) = e^t \cdot x \in C^2(\mathbb{R}^2)$:

$$\begin{aligned} dY_t &= \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial x} dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dX_t \cdot dX_t) \\ &= Y_t dt + e^t dX_t + 0 \cdot (dX_t \cdot dX_t). \end{aligned}$$

Now, substituting in (11) for dX_t, we obtain:

$$= Y_t dt + e^t[(m - X_t)dt + \sigma dW_t]$$

$$= (Y_t + e^t(m - X_t))dt + \sigma e^t dW_t = e^t m dt + \sigma e^t dW_t$$

Removing the dependence on Y_t on the right-hand side enables us to solve the above simplified SDE to obtain:

$$Y_t = Y_0 + \int_0^t m e^t dt + \sigma \int_0^t e^s dW_s$$

with $Y_0 = g(0, X_0) = g(0, x) = x$ giving

$$X_t = x e^{-t} + m(1 - e^{-t}) + \sigma \int_0^t e^{-(t-s)} dW_s$$

Part (ii)

Assuming x is deterministic, we compute the mean and variance of X_t:

$$\mathbb{E}[X_t] = \mathbb{E}[x e^{-t}] + \mathbb{E}[m(1 - e^{-t})] + \mathbb{E}\left[\sigma \int_0^t e^{-(t-s)} dW_s\right]$$

by linearity. Since the first two terms above contain deterministic expressions and the last is the expected value of an Itô integral (in particular, the Itô integral of a smooth deterministic function), it has to mean zero. Thus,

$$\mathbb{E}[X_t] = x e^{-t} + m(1 - e^{-t}) = m + e^{-t}(x - m)$$

Now, computing the second moment of X_t:

$$\mathbb{E}[X_t^2] = \mathbb{E}\left[\left(x e^{-t} + m(1 - e^{-t})\right)^2\right] + \mathbb{E}\left[2\left(x e^{-t} + m(1 - e^{-t})\right) \cdot \int_0^t \sigma e^{-(t-s)} dW_s\right]$$

$$+\mathbb{E}\left[\left(\int_0^t \sigma e^{-(t-s)} dW_s\right)^2\right]$$

$$= \left(xe^{-t} + m(1-e^{-t})\right)^2 + 2\left(xe^{-t} + m(1-e^{-t})\right)\mathbb{E}\left[\int_0^t \sigma e^{-(t-s)} dW_s\right] + \mathbb{E}\left[\left(\int_0^t \sigma e^{-(t-s)} dW_s\right)^2\right]$$

$$= \mathbb{E}[X_t]^2 + 0 + \mathbb{E}\left[\int_0^t \sigma^2 e^{-2(t-s)} ds\right]$$

where linearity of expectation, Itô Isometry and the mean zero property of Itô integrals for sufficiently regular functions (i.e. progressively measurable and in $L^2([0,t]\times\Omega)$) were used. Thus the variance is computed easily as:

$$\mathrm{Var}[X_t] = \mathbb{E}[X_t^2] - \mathbb{E}[X_t]^2$$

$$= \mathbb{E}\left[\int_0^t \sigma^2 e^{-2(t-s)} ds\right]$$

$$= \int_0^t \sigma^2 e^{-2(t-s)} ds = e^{-2t} \int_0^t \sigma^2 e^{2s} ds = \frac{\sigma^2}{2}(1 - e^{-2t})$$

Part (iii)

By part (ii), the equation for the second moment is:

$$\mathbb{E}[X_t^2] = \frac{\sigma^2}{2}(1 - e^{-2t}) + \mathbb{E}[X_t]^2$$

$$= \frac{\sigma^2}{2}(1 - e^{-2t}) + \left(\mathbb{E}[X_0]e^{-t} + m(1 - e^{-t})\right)^2$$

here we allow for the case that X_0 is random, but independent of the Brownian motion W_t driving (11)

Part (iv)

We first note that X_t is a Gaussian process since for all t, X_t is the sum of a deterministic part and an Itô integral of a deterministic function, which has a Gaussian distribution. This follows as the integral above is defined as an L^2 limit of integrals of simple functions, which palpably are Gaussian, being linear combinations of independent Gaussians. This means that

$$X_t \sim \mathcal{N}\left(m + e^{-t}(x - m), \frac{\sigma^2}{2}(1 - e^{-2t})\right)$$

Asymptotically, as $t \to \infty$, the mean and variance visibly converge to $m, \frac{\sigma^2}{2}$. This means that the first and second moments of X_t, converge to the values above, yielding the following convergence in distribution;

$$X_t \xrightarrow{d} \mathcal{N}\left(m, \frac{\sigma^2}{2}\right).$$

Question 2

Part (a)

*ai)
The generator of the Fokker-Planck equation corresponding to this linear SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x \tag{12}$$

where $b(X_t) = \mu X_t$ and $\sigma(X_t) = \sigma X_t$, is:

$$\mathcal{L}^*(\cdot) = \boldsymbol{\nabla} \cdot (-\boldsymbol{b}(x) \cdot + \frac{1}{2}\boldsymbol{\nabla} \cdot (\Sigma \cdot))$$

with $\Sigma(x) = \sigma(x)\sigma(x)^T = \sigma^2 x^2$. This simplifies to

$$\mathcal{L}^*(\cdot) = -\frac{d}{dx}(\boldsymbol{b}(x)\cdot) + \frac{\sigma^2}{2}\frac{d^2}{dx^2}(x^2\cdot))$$
$$= -\frac{d}{dx}(\boldsymbol{\mu} x\cdot) + \frac{\sigma^2}{2}\frac{d^2}{dx^2}(x^2\cdot))$$

*aii)
Consider for $n \geq 1, n \in \mathbb{N}$, $M_t^n = (X_t)^n$, where X_t is a solution to (12). Applying Itô's formula to $g_n(t,x) = x^n \in C^2$ gives:

$$d(M_t^n) = d((X_t)^n) = nX_t^{n-1}dX_t + \frac{1}{2}n(n-1)X_t^{n-2}(dX_t \cdot dX_t)$$

Now, the last term being a product of two Itô differentials, we obtain

$$d(M_t^n) = d((X_t)^n) = nX_t^{n-1}(\mu X_t dt + \sigma X_t dW_t) + \frac{1}{2}n(n-1)X_t^{n-2}(\sigma^2 X_t^2)dt$$
$$= nM_t^n(\mu dt + \sigma dW_t) + \frac{\sigma^2}{2}n(n-1)M_t^n dt$$
$$= \left(n\mu M_t^n + \frac{\sigma^2}{2}n(n-1)M_t^n\right)dt + n\sigma M_t^n dW_t$$

Thus, in integral form, the nth moments satisfy:

$$M_t^n = M_0^n + \int_0^t \left(n\mu M_s^n + \frac{\sigma^2}{2}n(n-1)M_s^n\right)ds + \int_0^t n\sigma M_s^n dW_s$$

where $M_0^n = X_0^n = x^n$. Now, taking expectations of both sides, we obtain

$$\mathbb{E}[X_t^n] = \mathbb{E}[M_t^n] = x^n + \mathbb{E}\left[\int_0^t \left(n\mu M_s^n + \frac{\sigma^2}{2}n(n-1)M_s^n\right)ds\right] + \mathbb{E}\left[\int_0^t n\sigma M_s^n dW_s\right]$$

using linearity of the expectation. Furthermore,

$$\mathbb{E}[X_t^n] = \mathbb{E}[x]^n + \int_0^t \mathbb{E}\left[n\mu M_s^n + \frac{\sigma^2}{2}n(n-1)M_s^n\right]ds + 0$$

where the expectation and the integration was exchanged, essentially an application of Fubini's Theorem and the last term vanishes as it is the expectation of an Itô integral of the process M_t^n, which is adapted to the filtration of Brownian motion and is progressively measurable. Now, the above further simplifies to

$$\mathbb{E}[X_t^n] = \mathbb{E}[x]^n + \int_0^t n\mu\mathbb{E}[X_s^n] + \frac{\sigma^2}{2}n(n-1)\mathbb{E}[X_s^n]ds$$

Differentiating both sides now gives,

$$\frac{d}{dt}\mathbb{E}[X_t^n] = \left(n\mu + \frac{\sigma^2}{2}n(n-1)\right)\mathbb{E}[X_t^n], \quad \mathbb{E}[X_0^n] = \mathbb{E}[x^n]$$

Solving this ODE gives

$$\mathbb{E}[X_t^n] = \mathbb{E}[x^n]\exp\left[\left(n\mu + \frac{\sigma^2}{2}n(n-1)\right)t\right]$$

*aiii) Suppose that X_t solves (12) and consider $g(t, X_t) = \log(X_t) \in C^2$, i.e. the natural logarithm of the process X_t. We now apply Itô's lemma to obtain:

$$\begin{aligned} d(\log(X_t)) = dg(t, X_t) &= \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dX_t \cdot dX_t) \\ &= \frac{1}{X_t}(\mu X_t dt + \sigma X_t dW_t) - \frac{\sigma^2}{2X_t^2}X_t^2 dt = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW_t \end{aligned}$$

The above is now a linear SDE with constant coefficients which can readily be solved (by inspection) to give:

$$\log(X_t) = \log(X_0) + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t$$

Thus, exponentiating both sides finally gives the solution to (12):

$$\begin{aligned} X_t &= X_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right] \\ &= x\exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right] \end{aligned}$$

*aiv)
We now consider the Stratonovich interpretation of (12), namely

$$dX_t = \mu X_t dt + \sigma X_t \circ dW_t, \tag{13}$$

From lectures, we apply the Stratonovich correction

$$\left(\frac{1}{2}(\sigma x)\frac{d}{dx}(\sigma x)\right)(X_t)$$

to obtain the following equivalent Itô SDE:

$$dX_t = \left(\mu + \frac{\sigma^2}{2}\right)X_t dt + \sigma X_t dW_t$$

This is the same SDE as (12), except with the change of $\mu \to \left(\mu + \frac{\sigma^2}{2}\right)$ in the drift term.

Part (b)

The SDE:

$$dX_t = f(t)X_t dt + h(t)X_t dW_t, \quad X_0 = x$$

for f, h continuous can similarly be solved by applying Itô 's lemma to the function $V(x,t) = log(x)$. The SDE that the process $V(X_t, t) = \log(X_t)$ solves is given by:

$$d(\log(X_t)) = dV(t, X_t) = \frac{\partial V}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} dX_i + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 V}{\partial x_i \partial x_j} dX_i \cdot dX_j$$

whence

$$\begin{aligned} d(\log(X_t)) &= \frac{1}{X_t}(f(t)X_t dt + h(t)X_t dW_t) - \frac{h^2(t)}{2X_t^2} X_t^2 dt \\ &= \left(f(t) - \frac{h^2(t)}{2}\right) dt + h(t) dW_t, \end{aligned}$$

which in integral form becomes

$$\log(X_t) = \log(X_0) + \int_0^t \left(f(s) - \frac{h^2(s)}{2}\right) ds + \int_0^t h(s) dW_s$$

and exponentiating both sides gives

$$\begin{aligned} X_t &= X_0 \exp\left[\int_0^t \left(f(s) - \frac{h^2(s)}{2}\right) ds + \int_0^t h(s) dW_s\right] \\ &= x \cdot \exp\left[\int_0^t \left(f(s) - \frac{h^2(s)}{2}\right) ds + \int_0^t h(s) dW_s\right] \end{aligned}$$

Part(c)

Suppose X_t solves the SDE

$$dX_t = b(X_t)dt + X_t dW_t$$

Again, we consider the twice-continuously differentiable transformation $\log(X_t)$. Using Itô's lemma, the transformed SDE for $\log(X_t)$ now becomes:

$$\begin{aligned} d(\log(X_t)) &= \frac{1}{X_t}(b(X_t)dt + X_t dW_t) - \frac{1}{2X_t^2} X_t^2 dt \\ &= \left(\frac{b(X_t)}{X_t} - \frac{1}{2}\right) dt + dW_t. \end{aligned}$$

Now, relabelling the process $Y_t = \log(X_t)$, we obtain:

$$dY_t = \left(e^{-Y_t} b(e^{Y_t}) - \frac{1}{2}\right) dt + dW_t.$$

The above being an SDE with additive noise as required.

Part (d)

Suppose now that X_t is a solution to

$$dX_t = \left(c + \frac{1}{2}\sum_{j=1}^{n} \alpha_j\right) X_t dt + X_t \sum_{j=1}^{n} \alpha_j dW_j(t)$$

Applying Itô 's lemma to the function $V(x,t) = \log(x)$. The SDE that the process $V(X_t, t) = \log(X_t)$ solves is given by:

$$d(\log(X_t)) = dV(t, X_t) = \frac{\partial V}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} dX_i + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 V}{\partial x_i \partial x_j} dX_i \cdot dX_j$$

yielding

$$d(\log(X_t)) = \frac{1}{X_t} dX_t - \frac{1}{2X_t^2}(dX_t \cdot dX_t)$$

Using the convention $dW_i \cdot dW_j = \delta_{ij} dt, dt \cdot dW_i = 0, dt \cdot dt = 0$, where $W_i, i = 1, \ldots, n$ are the components of an n-dimensional Brownian motion, $dX_t \cdot dX_t$ becomes:

$$dX_t \cdot dX_t = X_t^2 \sum_{j=1}^{n} \alpha_j^2 dt$$

The convention can be justified since

$$\mathbb{E}[(dt \cdot dW_i)^2]^{\frac{1}{2}} = dt^{\frac{3}{2}}$$

which can be ignored when considering first-order terms in dt. Hence,

$$\begin{aligned} d(\log(X_t)) &= \left(c + \frac{1}{2}\sum_{j=1}^{n} \alpha_j\right) dt + \sum_{j=1}^{n} \alpha_j dW_j(t) - \frac{1}{2}\sum_{j=1}^{n} \alpha_j^2 dt \\ &= cdt + \sum_{j=1}^{n} \alpha_j dW_j(t) \end{aligned}$$

an SDE with additive noise. In integral form, this yields:

$$\log(X_t) = \log(X_0) + ct + \sum_{j=1}^{n} \alpha_j W_j(t)$$

Thus, the solution to the SDE is

$$\begin{aligned} X_t &= X_0 \cdot \exp\left[ct + \sum_{j=1}^{n} \alpha_j W_j(t)\right] \\ &= x \cdot \exp\left[ct + \sum_{j=1}^{n} \alpha_j W_j(t)\right]. \end{aligned}$$

as $Y_0 = \sqrt{(}X_0) = \sqrt{(}x)$.

Question 3

Part (a)

Applying Itô 's lemma (1) to the smooth transformation $X_t = x^2$ of W_t, a standard one-dimensional Brownian motion, gives

$$dX_t = 2W_t dW_t + \frac{1}{2}2dt$$

$$= 2W_t dW_t + dt = dt + 2\sqrt{X_t}dW_t$$

and also observe that $X_0 = W_0^2 = 0$ almost surely as W_t is a Brownian motion. Thus, X_t indeed solves the SDE in question.

Part (b)

Suppose that X_t is a solution to the SDE:

$$dX_t = \left(\frac{1}{4} - X_t\right)dt + \sqrt{X_t}dW_t, \quad X_0 = x.$$

Consider the twice-continuously differentiable transformation $Y_t = \sqrt{X_t}$, from Itô 's lemma (1) with $V(t,x) = \sqrt{x}$, the process Y_t satisfies the SDE

$$dY_t = \frac{dV}{dx}dX_t + \frac{1}{2}\frac{d^2V}{dx^2}(dX_t \cdot dX_t)$$

$$= \frac{dV}{dx}(t, X_t)dX_t + \frac{1}{2}\frac{d^2V}{dx^2}(t, X_t)X_t dt$$

$$= \frac{1}{2\sqrt{X_t}}\left[\left(\frac{1}{4} - X_t\right)dt + \sqrt{X_t}dW_t\right] + \frac{1}{2}\left(-\frac{1}{4(X_t)^{\frac{3}{2}}}\right)X_t dt$$

$$= -\frac{1}{2\sqrt{X_t}}X_t dt + \frac{1}{2}dW_t$$

$$= -\frac{1}{2}Y_t dt + \frac{1}{2}dW_t$$

This SDE, can be by applying Itô 's lemma to the process $Z_t = e^{\frac{t}{2}}Y_t$, i.e. using (1) with $V(t,y) = e^{\frac{t}{2}}y$ yields the SDE

$$dZ_t = dV(t, Y_t) = \frac{\partial V}{\partial t}(t, Y_t)dt + \frac{\partial V}{\partial y}(t, Y_t)dY_t + \frac{1}{2}\frac{\partial^2 V}{\partial y^2}(t, Y_t)(dY_t \cdot dY_t)$$

$$= \frac{1}{2}e^{\frac{t}{2}}Y_t dt + e^{\frac{t}{2}}dY_t = \frac{1}{2}e^{\frac{t}{2}}Y_t dt + e^{\frac{t}{2}}\left(-\frac{1}{2}Y_t dt + \frac{1}{2}dW_t\right)$$

$$= \frac{e^{\frac{t}{2}}}{2}dW_t$$

In integral form, this becomes

$$e^{\frac{t}{2}}Y_t = Z_t = Z_0 + \int_0^t \frac{e^{\frac{s}{2}}}{2} dW_s = Y_0 + \int_0^t \frac{e^{\frac{s}{2}}}{2} dW_s$$

Thus,

$$X_t = Y_t^2 = \left(\sqrt{x} e^{-\frac{t}{2}} + \int_0^t \frac{e^{\frac{-(t-s)}{2}}}{2} dW_s \right)^2 .$$

Question 4

Part (a)

By definition, Y_1, Y_2 are:

$$Y_1(t) = \Re h(W_t), \quad Y_2(t) = \Im h(W_t),$$

where $h(x) = e^{ix}$. Now, Itô 's lemma (1), applied to $V_1(t,x) = \Re h(x)$ and $V_2(t,x) = \Im h(x)$ gives

$$\begin{cases} dY_1(t) = dV_1(t, W_t) = \frac{\partial V_1}{\partial t}(t, W_t)dt + \frac{\partial V_1}{\partial x}(t, W_t)dW_t + \frac{1}{2}\frac{\partial^2 V_1}{\partial x^2}(t, W_t)(dW_t \cdot dW_t) \\ dY_2(t) = dV_2(t, W_t) = \frac{\partial V_2}{\partial t}(t, W_t)dt + \frac{\partial V_2}{\partial x}(t, W_t)dW_t + \frac{1}{2}\frac{\partial^2 V_2}{\partial x^2}(t, W_t)(dW_t \cdot dW_t) \end{cases}$$

which further simplifies to

$$\begin{cases} dY_1(t) = -Y_2(t)dW_t - \frac{1}{2}Y_1(t)dt, \quad Y_1(0) = \Re h(W_0) = \Re h(0) = 1 \\ dY_2(t) = Y_1(t)dW_t - \frac{1}{2}Y_2(t)dt, \quad Y_2(0) = \Im h(W_0) = \Im h(0) = 0 \end{cases} \tag{14}$$

by virtue of the fact that

$$\frac{dh(x)}{dx} = ih(x), \quad \frac{d^2h(x)}{dx^2} = -h(x)$$

yielding

$$\frac{d\Re h(W_t)}{dx} = -\Im h(W_t), \quad \frac{d^2\Re h(W_t)}{dx^2} = -\Re h(W_t)$$

and

$$\frac{d\Im h(W_t)}{dx} = \Re h(W_t), \quad \frac{d^2\Im h(W_t)}{dx^2} = -\Im h(W_t).$$

Part (b)

The above system can be written more compactly as

$$d\boldsymbol{Y}_t = \boldsymbol{b}(\boldsymbol{Y}_t)dt + \boldsymbol{\sigma}(\boldsymbol{Y}_t)dW_t, \quad \boldsymbol{Y}_0 = \boldsymbol{y} \tag{15}$$

where

$$\boldsymbol{b}(\boldsymbol{Y}_t) = \begin{bmatrix} -\frac{1}{2}Y_1(t) \\ -\frac{1}{2}Y_2(t) \end{bmatrix}, \quad \boldsymbol{\sigma}(\boldsymbol{Y}_t) = \begin{bmatrix} -Y_2(t) \\ Y_1(t) \end{bmatrix}, \quad \boldsymbol{Y}_t = \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The generator of the above SDE (18) is given by

$$\mathcal{L} = \boldsymbol{b}(y)\cdot\nabla + \frac{1}{2}\Sigma(y) : D^2$$

$$= -\frac{1}{2}y_1\frac{\partial}{\partial y_1} - \frac{1}{2}y_2\frac{\partial}{\partial y_2} + \sum_{i,j=1}^{2}\Sigma_{ij}\frac{\partial^2}{\partial y_i \partial y_j}$$

where

$$\Sigma = \boldsymbol{\sigma}\boldsymbol{\sigma}^T = \begin{bmatrix} y_2^2 & -y_1y_2 \\ -y_1y_2 & y_1^2 \end{bmatrix}.$$

Thus, the generator is

$$\mathcal{L} = -\frac{1}{2}y_1\frac{\partial}{\partial y_1} - \frac{1}{2}y_2\frac{\partial}{\partial y_2} + \frac{y_2^2}{2}\frac{\partial^2}{\partial y_1\partial y_1} - y_1y_2\frac{\partial^2}{\partial y_1\partial y_2} + \frac{y_1^2}{2}\frac{\partial^2}{\partial y_2\partial y_2}$$

Part (c)

Consider the processes $X_1(t) = V_1(t, W_t) = \cos(W_t), X_2(t) = V_2(t, W_t) = \sin(W_t)$. Since these processes are smooth transformations of the same standard Brownian motion, Itô 's lemma applies yielding the following system

$$\begin{cases} dX_1(t) = \frac{\partial V_1}{\partial t}(t, W_t)dt + \frac{\partial V_1}{\partial x}(t, W_t)dW_t + \frac{1}{2}\frac{\partial^2 V_1}{\partial x^2}(t, W_t)(dW_t \cdot dW_t) \\ dX_2(t) = \frac{\partial V_1}{\partial t}(t, W_t)dt + \frac{\partial V_1}{\partial x}(t, W_t)dW_t + \frac{1}{2}\frac{\partial^2 V_1}{\partial x^2}(t, W_t)(dW_t \cdot dW_t) \end{cases}$$

$$\begin{cases} dX_1(t) = -X_2(t)dW_t - \frac{1}{2}X_1(t)dt, \quad X_1(0) = 1 \\ dX_2(t) = X_1(t)dW_t - \frac{1}{2}X_2(t)dt, \quad X_2(0) = 0 \end{cases}$$

which is the same SDE as (14). Since the above system is a lienar SDE satisfying the conditions

$$|\boldsymbol{b}(x)| + |\boldsymbol{\sigma}(x)| \le (1 + |x|), \quad x \in \mathbb{R}^2$$

and the initial condition $\boldsymbol{y}$ is deterministic implying boundedness of its second moment, by the Existence and Uniqueness theorem for SDE's,

$$X_t = \begin{bmatrix} \cos(W_t) \\ \sin(W_t) \end{bmatrix} = \boldsymbol{Y}_t$$

for all t almost surely. Thus,

$$Y_1(t) = \cos(W_t), \quad Y_2(t) = \sin(W_t)$$

for all t almost surely.

Question 5

Part (i)

We consider the Stratonovich SDE

$$dX_t = f(X_t) \circ dW_t, \quad X_0 = x \tag{16}$$

where f is positive and differentiable. From lectures, this SDE is equivalent to the following Itô SDE

$$dX_t = \frac{1}{2} f(X_t) f'(X_t) dt + f(X_t) dW_t$$

Now, define

$$h(x) = \int_0^x \frac{1}{f(z)} dz$$

which is well-defined by the positivity of $f(x)$. Also, by the fundamental theorem of calculus, and the differentiability of f, $g(t,x)$ is twice differentiable. Applying Itô 's lemma (1) with $V(t,x) = h(x)$ gives

$$dh(X_t) = \frac{dh}{dx} dX_t + \frac{1}{2} \frac{d^2h}{dx^2} (dX_t \cdot dX_t)$$

$$\frac{dh}{dx}\left(\frac{1}{2} f(X_t) f'(X_t) dt + f(X_t) dW_t\right) + \frac{1}{2}\frac{d^2h}{dx^2} f^2(X_t) dt$$

But, by the Fundamental theorem of calculus, we have

$$\frac{dh}{dx} = \frac{1}{f(x)}, \quad , \frac{d^2h}{dx^2} = -\frac{f'(x)}{f^2(x)}.$$

Thus,

$$\begin{aligned} dh(X_t) &= \frac{1}{f(x)}\left(\frac{1}{2} f(X_t) f'(X_t) dt + f(X_t) dW_t\right) - \frac{f'(x)}{2f^2(x)} f^2(X_t) dt \\ &= \left(\frac{1}{2} f'(X_t) dt + dW_t\right) - \frac{f'(x)}{2} dt = dW_t \end{aligned}$$

This yields that $h(X_t) = h(X_0) + W_t = h(x) + W_t$. Since $f(x)$ is positive, it follows that $\frac{dh}{dx} > 0$, meaning $h(x)$ is both strictly increasing and differentiable, hence invertible. This allows us to express X_t as

$$X_t = h^{-1}(h(X_t)) = h^{-1}(h(x) + W_t).$$

Part (ii)

Now, to solve the SDE

$$dX_t = -\frac{1}{2} X_t dt + \sqrt{1 - X_t^2} dW_t, \quad X_0 = x \in [-1, 1],$$

we notice that it is equivalent to the Stratonovich SDE (16) with $f(x) = \sqrt{1-x^2}$, which is non-negative and differentiable, since the drift term

$$-\frac{1}{2}X_t dt$$

is precisely the Stratonovich correction term

$$\frac{1}{2}f(X_t)f'(X_t)dt$$

with f as above. Thus, by part (i), the solution to the above SDE is given by

$$X_t = h^{-1}(h(x) + W_t)$$

where

$$h(x) = \int_0^x \frac{1}{f(z)}dz = \int_0^x \frac{1}{\sqrt{1-z^2}}dz = \arcsin(x)$$

Finally giving

$$X_t = h^{-1}(h(x) + W_t) = \sin(\arcsin(x) + W_t).$$

Question 6

In this question, we consider the following system of SDEs:

$$\begin{cases} dq_t = p_t dt, \quad q_0 = q, \\ dp_t = -q_t dt + z_t dt, \quad p_0 = p, \\ dz_t = -z_t dt - p_t dt + \sqrt{2}dW_t, \quad z_0 \sim \mathcal{N}(0,1) \end{cases} \tag{17}$$

part (i)

The above system can be written more compactly as

$$d\boldsymbol{X}_t = \boldsymbol{b}(\boldsymbol{X}_t)dt + \boldsymbol{\sigma}(\boldsymbol{X}_t)dW_t \tag{18}$$

where

$$\boldsymbol{b}(x) = \begin{bmatrix} x_2 \\ -x_1 + x_3 \\ -x_3 - x_2 \end{bmatrix}, \quad \boldsymbol{\sigma}(x) = \begin{bmatrix} 0 \\ 0 \\ \sqrt{2} \end{bmatrix}, \quad \boldsymbol{X}_t = \begin{bmatrix} q_t \\ p_t \\ z_t \end{bmatrix}$$

The generator of the above SDE (18) is given by

$$\mathcal{L} = \boldsymbol{b}(x) \cdot \nabla + \frac{1}{2}\Sigma(x) : D^2$$

$$= x_2 \frac{\partial}{\partial x_1} + (-x_1 + x_3)\frac{\partial}{\partial x_2} - (x_3 + x_2)\frac{\partial}{\partial x_3} + \frac{1}{2}\sum_{i,j=1}^{3} \Sigma_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

where

$$\Sigma = \boldsymbol{\sigma}\boldsymbol{\sigma}^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Thus, the generator is

$$\mathcal{L} = x_2 \frac{\partial}{\partial x_1} + (-x_1 + x_3)\frac{\partial}{\partial x_2} - (x_3 + x_2)\frac{\partial}{\partial x_3} + \frac{\partial^2}{\partial x_3 \partial x_3}$$

Part (ii)

Define the function $g(t,x) = e^t x_3$. Now, applying Itô 's lemma (1) to $g(t, \boldsymbol{X}_t)$ gives:

$$\begin{aligned} dg(t,\boldsymbol{X}_t) &= \frac{\partial g}{\partial t}(t,\boldsymbol{X}_t)dt + \mathcal{L}g(t,\boldsymbol{X}_t)dt + \langle \nabla g(t,\boldsymbol{X}_t), \boldsymbol{\sigma}(\boldsymbol{X}_t)dW_t \rangle \\ &= g(t,\boldsymbol{X}_t)dt - (z_t + p_t)e^t dt + \sqrt{2}e^t dW_t \\ &= -p_t e^t dt + \sqrt{2}e^t dW_t \end{aligned}$$

In integral form, this reads

$$g(t,\boldsymbol{X}_t) = g(0,\boldsymbol{X}_0) - \int_0^t p_s e^s ds + \sqrt{2}\int_0^t e^s dW_s$$

or equivalently,

$$z_t = z_0 - \int_0^t p_s e^{-(t-s)} ds + \sqrt{2}\int_0^t e^{-(t-s)} dW_s$$

In view of the above, we obtain the following for q_t, p_t in (17):

$$\begin{cases} \frac{dq_t}{dt} = p_t \\ \frac{dp_t}{dt} = -q_t - \int_0^t p_s e^{-(t-s)} ds + F(t) \end{cases} \tag{19}$$

where the process $F(t)$ is given by

$$F(t) = z_0 e^{-t} + \sqrt{2}\int_0^t e^{-(t-s)} dW_s \tag{20}$$

We now investigate the properties of $F(t)$. First, we see that it is mean zero, as

$$\mathbb{E}[F(t)] = \mathbb{E}[z_0 e^{-t}] + \mathbb{E}\left[\sqrt{2}\int_0^t e^{-(t-s)} dW_s\right] = 0$$

by linearity of expectation and that $z_0 \sim \mathcal{N}(0,1)$ and the second term is an Itô integral of a smooth deterministic function (is adapted and measurable). To compute the auto-correlation function, we use Itô 's isometry to obtain

$$\mathbb{E}[F(s)F(t)] = \mathbb{E}\left[\left(z_0 e^{-t} + \sqrt{2}\int_0^t e^{-(t-s)} dW_s\right)\left(z_0 e^{-s} + \sqrt{2}\int_0^s e^{-(s-r)} dW_r\right)\right]$$

$$= \mathbb{E}[z_0^2]e^{-(s+t)} + 2 \cdot \mathbb{E}\left[\int_0^t e^{-(t-u)} dW_u \int_0^s e^{-(s-v)} dW_v\right]$$

where the independence of z_0 from W_t allowed us to factorise the cross terms and use the mean zero property of the Itô integral. Without loss of generality, let $t > s$. Now, we compute the second term:

$$\mathbb{E}\left[\int_0^t e^{-(t-u)} dW_u \int_0^s e^{-(s-v)} dW_v\right] = \mathbb{E}\left[\int_0^s e^{-(t-u)} dW_u \int_0^s e^{-(s-v)} dW_v\right]$$

$$+\mathbb{E}\left[\int_s^t e^{-(t-u)} dW_u \int_0^s e^{-(s-v)} dW_v\right]$$

$$= \mathbb{E}\left[\int_0^s e^{-(s+t)} e^{2u} du\right] = e^{-(s+t)} \int_0^s e^{2u} du = \frac{1}{2} e^{-(s+t)} \left(e^{2s} - 1\right) = \frac{1}{2}\left(e^{(s-t)} - e^{-(s+t)}\right).$$

By an extension of the Itô isometry to products of progressively measurable and square-integrable functions. The second term vanishes since the simple approximations to the integrals (of deterministic functions)

$$I_1 = \int_s^t e^{-(t-u)} dW_u, \quad I_2 = \int_0^s e^{-(s-v)} dW_v$$

involve sums of increments of the Brownian motion W_t over intervals of disjoint interiors. Thus, by independence all terms vanish when taking the expectation of the product

$$0 = \mathbb{E}[I_{1n} I_{2n}] \to \mathbb{E}[I_1 I_2]$$

where $I_{1n}, I_{2n} \to I_1, I_2$ respectively in $L^2(\mathbb{P})$, are integrals of simple approximations to the integrands. Thus,

$$\mathbb{E}[F(s)F(t)] = e^{-(s+t)} + \left(e^{(s-t)} - e^{-(s+t)}\right) = e^{(s-t)} = e^{-|t-s|}$$

as we assumed $t > s$. By symmetry, the case for $t \leq s$ is nearly identical, and we obtain that the auto-correlation function is

$$\mathbb{E}[F(s)F(t)] = e^{-|t-s|}$$

as required. Note from (20) that the process $F(t)$ as defined is a Gaussian process, being a sum of the Itô integral of a deterministic function and the normal random variable $z_0 e^{-t}$ that is independent of the Brownian motion defining the integral (Gaussian random variables are closed under $L^2(\mathbb{P})$ limits). This is because the sum of two independent Gaussians is a Gaussian random variable.

Stationarity now follows because the auto-correlation function and the mean of the process being always zero, both only depend on $|s - t|$,
the relative separation in time and the fact that the auto-correlation function and the mean are sufficient to determine finite dimensional joint distributions since $F(t)$ is a Gaussian process.

Part (iii)

By inspection of the system derived for the process (q_t, p_t) (19), we notice that the derivative of p_t depends on the entire history of $p_s, s \leq t$ in a non-trivial way. Hence, we conclude that (q_t, p_t) is **not** a Markov process [2].

However, we note that the system (17) is a linear sde with constant coefficients, additive noise and only depends on the present state of (q_t, p_t, z_t). Thus, by a result from lectures, it has a strong solution that is a Markov process.

Question 7

Part (i)

Suppose X_t satisfies the SDE

$$dX_t = f(X_t)dt + g(X_t)dW_t \tag{21}$$

Let $V(t,x) = e^{\theta B(x)}$. Assuming $V(t,x)$ is sufficiently regular to apply Itô 's lemma, we have

$$\begin{aligned}
dV(t,X_t) &= \frac{\partial V}{\partial t}(t,X_t)dt + \frac{\partial V}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(t,X_t)(dX_t \cdot dX_t) \\
&= \theta B'(X_t)e^{\theta B(X_t)}dX_t + \frac{1}{2}\Big(\theta B''(X_t)e^{\theta B(X_t)} + \theta^2 B'(X_t)^2 e^{\theta B(X_t)}\Big)(dX_t \cdot dX_t) \\
&= \theta B'(X_t)e^{\theta B(X_t)}\Big(f(X_t)dt + g(X_t)dW_t\Big) \\
&\quad + \frac{1}{2}\Big(\theta B''(X_t)e^{\theta B(X_t)} + \theta^2 B'(X_t)^2 e^{\theta B(X_t)}\Big)g^2(X_t)dt
\end{aligned}$$

Now, we note that

$$B'(x) = \frac{dB}{dx} = \frac{1}{g(x)}, B''(x) = \frac{d^2B}{dx^2} = -\frac{g'(x)}{g^2(x)}.$$

Substituting the above, we obtain:

$$\begin{aligned}
dV(t,X_t) &= \theta e^{\theta B(X_t)}\Big(\frac{f(X_t)}{g(X_t)}dt + dW_t\Big) + \frac{1}{2}\Big(-\theta g'(X_t)e^{\theta B(X_t)} + \theta^2 e^{\theta B(X_t)}\Big)dt \\
&= e^{\theta B(X_t)}\Big(\theta\frac{f(X_t)}{g(X_t)} - \frac{1}{2}\theta g'(X_t) + \frac{\theta^2}{2}\Big)dt + \theta e^{\theta B(X_t)}dW_t \\
&= e^{\theta B(X_t)}\Big(\theta Z(X_t) + \frac{\theta^2}{2}\Big)dt + \theta e^{\theta B(X_t)}dW_t
\end{aligned}$$

where

$$Z(x) = \frac{f(x)}{g(x)} - \frac{1}{2}\frac{dg}{dx}(x)$$

Now, since

$$\theta = -\frac{1}{\frac{dZ}{dx}}\frac{d}{dx}\left(g(x)\frac{dZ}{dx}\right)$$

we can rearrange and integrate both sides to obtain

$$\frac{dZ}{dx} + \frac{\theta}{g(x)} Z(x) = \frac{A}{g(x)}$$

where A is some constant. We notice that this is a first order linear ODE and is readily solved by multiplying both sides by the integrating factor

$$\exp\left[\theta \int^x \frac{1}{g(s)} ds\right] = V(x)$$

which gives

$$\frac{d}{dx}(Z(x)V(x)) = \frac{A}{\theta}\frac{\theta}{g(x)}V(x) = \frac{A}{\theta}\frac{dV}{dx}$$

(note $\theta \neq 0$), implying that

$$Z(x) = \frac{A}{\theta} + \frac{B}{V(x)} \tag{22}$$

for constants A, B. Returning to the transformed SDE for $V(t, X_t)$ and substituting for $Z(X_t)$:

$$dV(t,X_t) = V(t,X_t)\Big(A + \frac{B\theta}{V(t,X_t)} + \frac{\theta^2}{2}\Big)dt + \theta V(t,X_t)dW_t$$

$$= \left[B\theta + V(t,X_t)\Big(A + \frac{\theta^2}{2}\Big)\right]dt + \theta V(t,X_t)dW_t$$

reducing the SDE (21) into a linear one as required.

Part (ii)

Now, we consider the SDE

$$dX_t = (\lambda X_t - X_t^2)dt + X_t dW_t \tag{23}$$

Setting $f(x) = \lambda x - x^2$ and $g(x) = x$, $Z(x)$ now becomes

$$Z(x) = \frac{f(x)}{g(x)} - \frac{1}{2}\frac{dg}{dx}(x) = \left(\lambda - \frac{1}{2}\right) - x.$$

Now, we also compute $\theta(x)$:

$$\theta(x) = -\frac{1}{\frac{dZ}{dx}}\frac{d}{dx}\left(g(x)\frac{dZ}{dx}\right) = \frac{d}{dx}(-x) = -1$$

a constant. Also, we obtain the transformed process $Y_t = V(t, X_t) = \exp(\theta B(X_t))$ with $B(x)$:

$$B(x) = \int_1^x \frac{1}{g(s)ds} = \int_1^x \frac{1}{s} ds = \log(x)$$

as

$$Y_t = V(t, X_t) = \exp(\theta B(X_t)) = \exp(-\log(x) = \frac{1}{X_t}$$

We notice that $Z(x)$ is of the desired form in (22) with $A = \left(\frac{1}{2} - \lambda\right), B = -1, V(x) = \frac{1}{x}$. Thus, the discussion in the previous part applies, that the process Y_t is the solution to the linear SDE

$$\begin{aligned} dY_t &= \left[B\theta + Y_t\left(A + \frac{\theta^2}{2}\right)\right]dt + \theta Y_t dW_t \\ &= \left[-B + Y_t\left(A + \frac{1}{2}\right)\right]dt - Y_t dW_t. \\ &= \left[1 + Y_t\left(1 - \lambda\right)\right]dt - Y_t dW_t. \end{aligned}$$

References

[1] Grigorios A Pavliotis. *Stochastic processes and applications: diffusion processes, the Fokker-Planck and Langevin equations*, volume 60, chapter 1,2,3, pages 67,68. Springer, 2014. pages 1

[2] Grigorios A Pavliotis. *Stochastic processes and applications: diffusion processes, the Fokker-Planck and Langevin equations*, volume 60, page 273. Springer, 2014. pages 19