Imperial College London

COURSEWORK 2

IMPERIAL COLLEGE LONDON

DEPARTMENT OF MATHEMATICS

MATH60029
Functional Analysis

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Problems

Problem set IV, exercise IV.4

Consider the linear map $T : \ell^2(\mathbb{R}) \to \ell^2(\mathbb{R})$, given by

$$T((x_n)_{n\geq 1}) = (x_1 + x_2, x_2 + x_3, \dots) \tag{1}$$

Show that it is bounded and compute its operator norm

$$\|T\|_{\mathcal{L}(\ell^2(\mathbb{R}),\ell^2(\mathbb{R}))} = \sup_{\substack{(x_n)_{n\geq 1}\in\ell^2(\mathbb{R}),\\ \|x\|_2\neq 0}} \frac{\|T((x_n)_{n\geq 1})\|_2}{\|(x_n)_{n\geq 1}\|_2} \tag{2}$$

Problem set VI, exercise VI.2

VI.2.i

Let X be a real normed space and M be a real vector subspace of X. Fix $y \in X \setminus \overline{M}$. Show that there exists a bounded linear functional $\ell \in X^*$ such that $\ell|_{\overline{M}} \equiv 0$ and

$$\|\ell^*\|_{\mathcal{L}(X,\mathbb{R})} = \frac{1}{\operatorname{dist}(y, M)}.$$

VI.2.ii

Let M be a closed subspace of the normed space X, then one needs to show that

$$M = \bigcap_{f\in X^*: M\subset \ker f} \ker f. \tag{3}$$

Problem set VII, exercise VII.5

Let X be a Banach space and let $T : X \to X$ be a linear operator such that for all $\phi \in X^*$, the map $\phi \circ T$ is continuous. Using the closed graph theorem to show that T is bounded.

Solutions

Problem set IV, exercise IV.4

Consider the map $T : \ell^2(\mathbb{R}) \to \ell^2(\mathbb{R})$, given by

$$T((x_n)_{n\geq 1}) = (x_1 + x_2, x_2 + x_3, \dots) \tag{4}$$

Linearity is easily verified since addition and multiplication of scalars is defined pointwise in $\ell^2(\mathbb{R})$. More explicitly, for all $(x_n)_{n\geq 1}, (y_n)_{n\geq 1} \in \ell^2(\mathbb{R})$:

$$T((x_n)_{n\geq 1} + (y_n)_{n\geq 1}) = T((x_n + y_n)_{n\geq 1}) = (x_1 + y_1 + x_2 + y_2, x_2 + y_2 + x_3 + y_3, \dots)$$

$$= (x_1 + x_2, x_2 + x_3, \dots) + (y_1 + y_2, y_2 + y_3, \dots) = T((x_n)_{n\geq 1}) + T((y_n)_{n\geq 1})$$

and for $\lambda \in \mathbb{R}$:

$$T(\lambda (x_n)_{n\geq 1}) = T((\lambda x_n)_{n\geq 1}) = (\lambda x_1 + \lambda x_2, \lambda x_2 + \lambda x_3, \dots) = \lambda (x_1 + x_2, x_2 + x_3, \dots) = \lambda \cdot T((x_n)_{n\geq 1})$$

For boundedness, it suffices to show that for $(x_n)_{n\geq 1} \in \ell^2(\mathbb{R})$ arbitrary

$$\|T((x_n)_{n\geq 1})\|_2 \leq C \cdot \|(x_n)_{n\geq 1}\|_2$$

for some $C \in \mathbb{R}$ independent thereof. Now let $(x_n)_{n\geq 1} \in \ell^2(\mathbb{R})$ be arbitrary. Then,

$$\|T((x_n)_{n\geq 1})\|_2^2 = \sum_{n\geq 1} (x_n + x_{n+1})^2$$

Now, since for all $x, y \in \mathbb{R}$

$$(x + y)^2 \leq 2(x^2 + y^2)$$

one has the following control over finite partial sums with $N \geq 1$:

$$\sum_{n=1}^{N} (x_n + x_{n+1})^2 \leq \sum_{n=1}^{N} 2(x_n^2 + x_{n+1}^2) \leq 2 \sum_{n\geq 1} (x_n^2 + x_{n+1}^2)$$

$$= 2 \sum_{n\geq 1} x_n^2 + \sum_{n\geq 1} x_{n+1}^2 \leq 2 \sum_{n\geq 1} x_n^2 + \sum_{n\geq 0} x_{n+1}^2 = 4 \sum_{n\geq 1} x_n^2 = 4 \cdot \|(x_n)_{n\geq 1}\|_2^2$$

Thus, taking $N \to \infty$, we have

$$\|T((x_n)_{n\geq 1})\|_2^2 \leq 4 \cdot \|(x_n)_{n\geq 1}\|_2^2$$

Equivalently,

$$\|T((x_n)_{n\geq 1})\|_2 \leq 2 \cdot \|(x_n)_{n\geq 1}\|_2 \quad \forall (x_n)_{n\geq 1} \in \ell^2(\mathbb{R}) \tag{5}$$

thereby showing boundedness with $C = 2$.

In fact we compute its operator norm to be

$$\|T\|_{\mathcal{L}(\ell^2(\mathbb{R}),\ell^2(\mathbb{R}))} = \sup_{\substack{(x_n)_{n\geq 1}\in\ell^2(\mathbb{R}),\\ \|x\|_2\neq 0}} \frac{\|T((x_n)_{n\geq 1})\|_2}{\|(x_n)_{n\geq 1}\|_2} = 2 \tag{6}$$

By (5),

$$\sup_{\substack{(x_n)_{n\geq 1}\in\ell^2(\mathbb{R}),\\ \|x\|_2\neq 0}} \frac{\|T((x_n)_{n\geq 1})\|_2}{\|(x_n)_{n\geq 1}\|_2} \leq 2$$

To show equality, consider the sequence

$$(x)_{k\geq 0} = \sum_{j=0}^{k} e_k \in \ell^2(\mathbb{R})$$

where $(e_k)_{k\geq 1}$ is the ususal Schauder basis for $\ell^2(\mathbb{R})$, with $(e_k)(n) = \delta_{kn}$, $n \geq 1$. One easily computes the norms:

$$\|x_k\|_2 = k^{\frac{1}{2}} > 0$$

$$\|T(x_k)\|_2 = \|(\underbrace{2, 2, \ldots, 2,}_{k-1} 1, 0, \ldots)\|_2 = [4(k-1)+1]^{\frac{1}{2}}$$

Thus,

$$\frac{\|T(x_k)\|_2}{\|x_k\|_2} = \frac{[4(k-1)+1]^{\frac{1}{2}}}{k^{\frac{1}{2}}} = \left[4 - \frac{4}{k} + \frac{1}{k}\right]^{\frac{1}{2}} \to 2, \quad k \to \infty$$

Thus, $\|T\|_{\mathcal{L}(\ell^2(\mathbb{R}),\ell^2(\mathbb{R}))} = 2$, as claimed.

Problem set VI, exercise VI.2

VI.2.i

Consider the linear functional $\ell : W \to \mathbb{R}$, with $W = \operatorname{span}\{M, y\}$ given by

$$\ell(x + ty) = t, \quad x \in M, y \in X \setminus \overline{M}, t \in \mathbb{R}$$

Linearity is clear since M is a real vector subspace space of X, and for $x_1, x_2 \in M, t_1, t_2 \in \mathbb{R}$ and $\lambda \in \mathbb{R}$:

$$\ell((x_1 + t_1 y) + (x_2 + t_2 y)) = t_1 + t_2 = \ell(x_1 + t_1 y) + \ell(x_2 + t_2 y)$$

and

$$\ell(\lambda(x_1 + t_1 y)) = \ell(\lambda x_1 + \lambda t_1 y)) = \lambda t_1 = \lambda \ell(x_1 + t_1 y)$$

Now, notice for $x \in M, t \in \mathbb{R} \setminus \{0\}$:

$$\|x + ty\| = |t| \cdot \|\frac{x}{t} + y\| \geq |t| \cdot \operatorname{dist}(y, M) > 0$$

since $\frac{x}{t} \in M$ and $\operatorname{dist}(y, M) > 0$ since $y \in X \setminus \overline{M}$. Thus,

$$|\ell(x + ty)| = |t| \leq \frac{1}{\operatorname{dist}(y, M)} \|x + ty\|, \quad x \in M, t \in \mathbb{R}$$

Now, since ℓ is dominated on W by the sub-linear functional $\frac{1}{\operatorname{dist}(y,M)} \| \cdot \|$, by the Hahn -Banach Theorem from lectures, ℓ has an extension ℓ^* on X with

$$\ell^*|_W = \ell \quad \text{and} \quad |\ell^*(z)| \leq \frac{1}{\operatorname{dist}(y, M)} \|z\|, \quad z \in X \tag{7}$$

In particular, it follows ℓ^* is a bounded linear functional with operator norm

$$\|\ell^*\|_{\mathcal{L}(X, \mathbb{R})} \leq \frac{1}{\operatorname{dist}(y, M)}$$

Now, by (7) for $z \in M$, one has

$$\ell^*(z) = \ell(z) = \ell(z + 0 \cdot y) = 0, \quad \text{and,} \quad \ell^*(y) = \ell(0 + 1 \cdot y) = 1$$

Finally, since

$$\operatorname{dist}(y, M) := \inf_{x \in M} \|x - y\|$$

choose a sequence $(x_n)_{n \geq 0} \subset M$ with

$$\|x_n - y\| \to \operatorname{dist}(y, M), \quad n \to \infty$$

Now, noting that $y \in X \setminus \overline{M}$, implies $\|x_n - y\| > 0$ for $n \in \mathbb{N}$, one computes

$$\frac{|\ell^*(x_n - y)|}{\|x_n - y\|} = \frac{1}{\|x_n - y\|} \to \frac{1}{\operatorname{dist}(y, M)}, \quad n \to \infty$$

establishing that indeed

$$\|\ell^*\|_{\mathcal{L}(X, \mathbb{R})} = \frac{1}{\operatorname{dist}(y, M)}$$

as required.

VI.2.ii

Restating the statement to be proven, let M be a closed subspace of X, then one needs to show that

$$M = \bigcap_{f \in X^*: M \subset \ker f} \ker f \tag{8}$$

The first inclusion

$$M \subseteq \bigcap_{f \in X^*: M \subset \ker f} \ker f$$

clearly holds since the intersection is taken over sets that include M. Now suppose for a contradiction that

$$M \subsetneq \bigcap_{f \in X^*: M \subset \ker f} \ker f$$

This means that there exists $x \in X \setminus M$ such that

$$x \in \ker f, \quad \forall f \in X^* : M \subset \ker f \tag{9}$$

Since M is closed, section () implies the existence of a bounded linear functional ϕ such that

$$\phi|_M \equiv 0, \quad \phi(x) = 1, \quad \text{and} \quad \|\phi\|_{\mathcal{L}(X,\mathbb{R})} = \frac{1}{\operatorname{dist}(x, M)}$$

But, (9) implies $x \in \ker\phi$, a contradiction to the above ($\phi(x) = 1 \neq 0$), whence the desired equality of sets is established.

Problem set VII, exercise VII.5

For the linear operator $T : X \to X$, let

$$G(T) = \{(x, Tx) : x\} \subseteq X \times X$$

denote its graph. We proceed to show that $G(T)$ is a closed subset of $X \times X$ under the norm $\|(x, y)\|_{X\times X} = \|x\|_X + \|y\|_X, x, y \in X$. Suppose now that $(z_n = (x_n, Tx_n))_{n\geq 0} \subseteq G(T)$ with

$$z_n \overset{\|\cdot\|_{X\times X}}{\longrightarrow} z = (x, y) \in X \times X, \quad n \to \infty, \quad x, y \in X$$

equivalently

$$x_n \overset{\|\cdot\|_X}{\longrightarrow} x, \quad Tx_n \overset{\|\cdot\|_X}{\longrightarrow} y$$

Now, fix $\phi \in X^*$, by assumption one has that $\phi \circ T$ is continuous on X, yielding

$$\phi(T(x_n)) = \phi \circ T(x_n) \to \phi \circ T(x) = \phi(T(x)), \quad n \to \infty$$

also, one has by the continuity of ϕ:

$$\phi(T(x_n)) \to \phi(y), \quad n \to \infty$$

Since X is a Banach space, limits of $\{\phi(T(x_n))\}_{n\geq 0}$ are unique giving

$$\phi(T(x)) = \phi(y) \tag{10}$$

Now, suppose for a contradiction that $Tx \neq y$. let $W = \text{span}\{Tx - y\} \neq \{0_X\} \subseteq X$. Define the linear functional $\ell : W \to \mathbb{R}$ by

$$\ell(\lambda(Tx - y)) = \lambda \cdot \|Tx - y\|_X, \quad \lambda \in \mathbb{R}$$

Clearly, ℓ is dominated by the sub-linear functional $\|\cdot\|_X$ on W. Thus, by an application of the Hahn-Banach Theorem from lectures, one obtains an extension $\ell^* : X \to \mathbb{R}$ such that

$$\ell^*|_W = \ell, \quad \text{and} \quad |\ell^*(z)| \leq \|z\|_X, \quad z \in X$$

Now, since it was assumed that $Tx \neq y$,

$$\ell^*(Tx - y) = \ell(Tx - y) = \|Tx - y\|_X > 0$$

But, since $\ell^* \in X^*$, (10) implies that

$$\ell^*(Tx) = \ell^*(y) \implies \ell^*(Tx - y) = 0$$

by linearity, a contradiction. Thus, $Tx = y$ implying that the graph $G(T)$ of T is closed. Since the domain of T, $\mathcal{D}(A) = X$, is closed (complete too), the Closed Graph Theorem from lectures implies that the operator $T : X \to X$ is indeed bounded, as desired.