Imperial College London

COURSEWORK 1

IMPERIAL COLLEGE LONDON

DEPARTMENT OF MATHEMATICS

MATH60029 Functional Analysis

Author: Pantelis Tassopoulos

Date: November 2022

Problems

Problem Set I: I.1.4

Let $\Omega = \{f \in C^{\omega}(\mathbb{K}) : \frac{d^2}{dz^2}f - \frac{d}{dz}f - 2z = 0\}$. Show that $(\Omega, \oplus, (\mathbb{K}, +, \cdot), \odot))$ is not a linear space, where $\oplus$ and $\odot$ are the usual addition and multiplication operations defined pointwise for functions and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Problem Set II: II.7

Show that in any metric space (X, ρ), we have that for all $z \in X$ the function

$$x \mapsto \rho(z, x) \tag{1}$$

is continuous.

Problem set III: III.5 Parts iii) and iv)

Part iii)

Fix $j \in \mathbb{N}$ abitrary and let $\Omega_j = \{x \in \ell_p : x_j = 0\}$. Show that Ω_j is closed in $(\ell_p, \|\cdot\|)$.

Part iv)

Let $\Omega = \left\{x \in \ell_p : \forall j \in \mathbb{N}, |x_j| \leq C \cdot j^{-\frac{2}{p}}\right\}$ for some $C \in (0, \infty)$. Show that Ω is closed in $(\ell_p, \|\cdot\|_p)$.

Solutions

Problem Set I: I.1.4

Consider $\Omega = \{f \in C^\omega(\mathbb{K}) : \frac{d^2}{dz^2}f - \frac{d}{dz}f - 2z = 0\}$. We claim that $(\Omega, \oplus, (\mathbb{K}, +, \cdot), \odot))$ is not a linear space, where $\oplus$ and $\odot$ are the usual addition and multiplication operations defined pointwise for functions and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Suppose for a contradiction, that $(\Omega, \oplus, (\mathbb{K}, +, \cdot), \odot))$ is a linear space. It is easy to show that the function $f(z) = -z^2 - 2z \in C^\omega$ satisfies

$$\frac{d^2}{dz^2}f - \frac{d}{dz}f - 2z = -2 + 2z + 2 - 2z = 0$$

implying that $f \in \Omega$. Thus, since Ω is non-empty, we take $f_1 = f \in \Omega$ and its additive inverse and obtain for the zero function ϕ,

$$\phi = f_1 \oplus (-f_1) \in \Omega$$

using the fact that $(\Omega, \oplus)$ is an Abelian group and $\oplus$ is defined pointwise using the usual addition on $\mathbb{K}$. However, it is clear that

$$\frac{d^2}{dz^2}\phi - \frac{d}{dz}\phi - 2z = -2z \neq 0$$

for $z = 1$ say. Thus, we have obtained the desired contradiction estabilishing that $\Omega = \{f \in C^\omega(\mathbb{K}) : \frac{d^2}{dz^2}f - \frac{d}{dz}f - 2z = 0\}$ is not a linear space.

Problem Set II: II.7

We need to show that in an arbitrary metric space (X, ρ), we have that for all $z \in X$ the function

$$x \mapsto \rho(z, x) \tag{2}$$

is continuous. We fix an arbitrary sequence $(x_n)_{n\in\mathbb{N}}$ such that $\rho(x_n, x) \to 0,$ as $n \to \infty$. By the triangle inequality we have

$$\rho(x,z) \leq \rho(x,y) + \rho(y,z), \quad \forall x,y,z \in X$$

$$\rho(x,z) - \rho(y,z) \leq \rho(x,y), \quad \forall x,y,z \in X$$

and by symmetry

$$\rho(z,y) - \rho(z,x) \leq \rho(x,y), \quad \forall x,y,z \in X$$

implying

$$|\rho(z,y) - \rho(x,z)| \leq \rho(x,y), \quad \forall x,y,z \in X$$

Now with x_n in place of y we obtain

$$0 \leq |\rho(x_n,z) - \rho(x,z)| \leq \rho(x_n,x) \to 0, \quad \text{as} \quad n \to \infty$$

Thus, (2) is sequentially continuous with z arbitry, hence continuous with respect to ρ, as required.

Problem set III: III.5 Parts iii) and iv)

0.1 Part iii)

Fix $j \in \mathbb{N}$ abitrary and let $\Omega_j = \{x \in \ell_p : x_j = 0\}$.

For closedness of Ω_j in $(\ell_p, \|\cdot\|)$, consider the sequence of points $(x_n)_{n\in\mathbb{N}} \subseteq \Omega_j$ with $x_n \to x \in \ell_p$ as $n \to \infty$. More precisely,

$$\|x_n - x\|_{\ell_p} = \left(\sum_{k\in\mathbb{N}} |x_{n,k} - x_k|^p\right)^{\frac{1}{p}} \to 0 \tag{3}$$

Since all the terms in the norm in (4) are non-negative, for all $N \in \mathbb{N}, N \geq j$:

$$|x_j| = |x_{n,j} - x_j| \leq \left(\sum_{0\leq k\leq N} |x_{n,k} - x_k|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k\in\mathbb{N}} |x_{n,k} - x_k|^p\right)^{\frac{1}{p}} \to 0, \quad \text{as} \quad n \to \infty$$

This yields that $x_j = 0$, meaning that $x \in \Omega_j$, showing that Ω_j is closed in ℓ_p.

For the second part of the question, we suppose for a contradiction that there exists a point $y \in \Omega_j$ and a ball $B(y,\delta) = \{x \in \ell_p : \|x-y\|_{\ell_p} < \delta\}, \delta > 0$ around y such that it is contained in Ω_j. Pick $\alpha = \alpha_\epsilon = (\alpha_n)_{n\in\mathbb{N}} \in \ell_p$ given by

$$\alpha_n = \frac{\epsilon^{\frac{1}{p}}}{2^{\frac{n}{p}}} + y_n, \quad n \in \mathbb{N}$$

with $\epsilon > 0$. Now,

$$\|\alpha - y\|_{\ell_p} = \left(\sum_{k\in\mathbb{N}} |\alpha_k - y_k|^p\right)^{\frac{1}{p}} = \left(\sum_{k\in\mathbb{N}} \left|\frac{\epsilon^{\frac{1}{p}}}{2^{\frac{k}{p}}} + y_k - y_k\right|^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{k\in\mathbb{N}} \left|\frac{\epsilon}{2^k}\right|\right)^{\frac{1}{p}} = \epsilon^{\frac{1}{p}} < \delta \quad \text{for} \quad 0 < \epsilon = \tilde{\epsilon} = \frac{\delta^p}{2} < \delta^p$$

say. Thus, we have that $\alpha = \alpha_{\tilde{\epsilon}} \in B(y,\delta) \subseteq \Omega_j$. But,

$$\alpha_j = \frac{\epsilon^{\frac{1}{p}}}{2^{\frac{j}{p}}} + y_j = \frac{\epsilon^{\frac{1}{p}}}{2^{\frac{j}{p}}} > 0$$

a contradiction and we are done.

Part iv)

Let $\Omega = \left\{ x \in \ell_p : \forall j \in \mathbb{N}, |x_j| \leq C \cdot j^{-\frac{2}{p}} \right\}$ for some $C \in (0, \infty)$.

For closedness of Ω in $(\ell_p, \|\cdot\|_p)$, consider the sequence of points $(x_n)_{n\in\mathbb{N}} \subseteq \Omega$ with $x_n \to x \in \ell_p$ as $n \to \infty$. Now, we have for all $j \in \mathbb{N}$ and $N \geq j \in \mathbb{N}$:

$$|x_{n,j} - x_j| \leq \left(\sum_{0\leq k\leq N} |x_{n,k} - x_k|^p \right)^{\frac{1}{p}} \leq \left(\sum_{k\in\mathbb{N}} |x_{n,k} - x_k|^p \right)^{\frac{1}{p}} \to 0, \quad \text{as} \quad n \to \infty \tag{4}$$

Fix $\epsilon > 0$. By (4), there exists an $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$ and $j \in \mathbb{N}$ we have

$$\big||x_{n,j}| - |x_j|\big| \leq |x_{n,j} - x_j| < \epsilon \tag{5}$$

Fix $n = N_0$. Now we have for all $j \in \mathbb{N}$,

$$\big||x_{N_0,j}| - |x_j|\big| \leq |x_{N_0,j} - x_j| < \epsilon$$

Since $x_{N_0} \in \Omega$ we have additionally that

$$|x_{N_0,j}| \leq C \cdot j^{-\frac{2}{p}}$$

for all $j \in \mathbb{N}$. Combining the above, we obtain by the reverse triangle inequality used in (5):

$$|x_j| \leq \epsilon + C \cdot j^{-\frac{2}{p}} \quad \forall j \in \mathbb{N}$$

Taking $\epsilon \to 0$ since it was arbitrary, we obtain that $\forall j \in \mathbb{N}, |x_j| \leq C \cdot j^{-\frac{2}{p}}$. Thus, $x \in \Omega$ yielding that Ω is closed in ℓ_p as required and we are done.