Lemma 9: Let f be a linear functional on a LCS (X,P). Then $f \in X^* \Leftrightarrow \ker f$ is closed.

Pf: $\Rightarrow$ $\ker(f) = f^{-1}(\{0\})$ is closed for continuous f.
$\Leftarrow$: If $\ker(f)=X$, then $f\equiv 0$ is continuous. Assume $\ker(f)\neq X$ & fix $x_0 \in X\setminus\ker(f)$. Since $X\setminus\ker(f)$ is open, $\exists \varepsilon>0$, $\exists p_1,\dots,p_k \in P$ s.t. $\{x\in X : p_i(x-x_0)<\varepsilon, 1\le i\le k\} \subset X\setminus\ker(f)$. Let $U=\{x\in X \mid p_i(x)<\varepsilon, 1\le i\le k\}$. Then U is a nbhd of 0 in X, and $(x_0+U)\cap\ker(f)=\emptyset$. Note that U is convex and, in the real case, symmetric ($x\in U \Rightarrow -x\in U$) or in the complex case, balanced ($x\in U$, $|\lambda|\le 1 \Rightarrow \lambda x\in U$), and hence so is $f(U)$ as f is linear. If $f(U)$ is not bounded, then $f(U)$ is the whole scalar field and hence so is $f(x_0+U)=f(x_0)+f(U) \ni 0$ ↯ as $0\notin f(x_0+U)$. So $\exists M>0$ s.t. $|f(x)|\le M$ $\forall x\in U$. So given $\varepsilon>0$, $\frac{\varepsilon}{M}U$ is a nbhd of 0 in X & $|f(x)|\le\varepsilon$ $\forall x\in\frac{\varepsilon}{M}U$, thus f is continuous at 0, hence everywhere. Thus $f\in X^*$.

Thm 11 (Hahn–Banach) Let (X,P) be a LCS.
(i) Given a subspace Y of X & $g\in Y^*$, $\exists f\in X^*$ s.t. $f|_Y=g$.
(ii) Given a closed subspace Y of X and $x_0\in X\setminus Y$, $\exists f\in X^*$ s.t. $f|_Y=0$ and $f(x_0)\neq 0$.

Remark: So X^* separates the points of X.

Proof (i) By Lemma 8, $\exists n\in\mathbb{N}$, $p_1,\dots,p_k\in P$, $C>0$ s.t. $\forall y\in Y$ $|g(y)|\le C\cdot\max_{1\le i\le k} p_i(y)$.
Let $p(x)=C\max_{1\le i\le k}p_i(x)$, $x\in X$. Then p is a seminorm on X & $|g(y)|\le p(y)$ $\forall y\in Y$. By the H-B, $\exists$ linear functional f on X s.t. $f|_Y=g$ and $|f(x)|\le p(x)$ $\forall x\in X$. By Lemma 8, $f\in X^*$.
(ii) Let $Z=\operatorname{span}(Y\cup\{x_0\})$ & define a linear functional g on Z by $g(y+\lambda x_0)=\lambda$, $y\in Y$, λ scalar. Then $g|_Y=0$, $g(x_0)=1\neq 0$ and $\ker(g)=Y$ is closed in Z. By Lemma 10, $g\in Z^*$. By part (i) $\exists f\in X^*$ s.t. $f|_Z=g$ and hence f works. □

2. Dual Spaces of $L_p(\mu)$ & $C(K)$

Let $(\Omega,\mathcal{F},\mu)$ be a measure space. For $1\le p<\infty$,
$$L_p(\mu)=\{f:\Omega\to\text{scalars} \mid f \text{ measurable}, \int_\Omega |f|^p\,d\mu<\infty\}$$
This is a normed space in the L_p norm $\|f\|_p=\left(\int|f|^p\,d\mu\right)^{1/p}$.

$p=\infty$: A measurable function $f:\Omega\to$ scalars is essentially bounded if $\exists N\in\mathcal{F}$, $\mu(N)=0$, and $f|_{\Omega\setminus N}$ is bounded.

$L_\infty(\mu)=\{f:\Omega\to\text{scalars} \mid f \text{ measurable \& essentially bounded}\}$

This is a normed space in the L_∞ norm
$\|f\|_\infty=\operatorname{ess\,sup}|f|=\inf\{\sup_{\Omega\setminus N}|f| \mid N\in\mathcal{F}, \mu(N)=0\}$.
The inf is attained: $\exists N\in\mathcal{F}$, $\mu(N)=0$, $\|f\|_\infty=\sup_{\Omega\setminus N}|f|$.

In all the cases, we identify functions f,g if $f=g$ a.e.

Thm 1: $L_p(\mu)$ is complete $1\le p\le\infty$. □

<u>Complex measures</u>: Let Ω be a set, $\mathcal{F}$ a σ-field on Ω. A complex measure on $\mathcal{F}$ is a countably additive function $\nu:\mathcal{F}\to\mathbb{C}$.
The total variation measure $|\nu|$ of ν is defined as follows:
$$|\nu(A)| = \sup\left\{\sum_{k=1}^{n}|\nu(A_k)| \;\middle|\; A=\bigcup_{k=1}^{n}A_k \text{ is a measurable partition of } A\right\}, \quad A\in\mathcal{F}$$
$A_k\in\mathcal{F}$, $A_j\cap A_k=\emptyset$ $\forall j\neq k$.
Then, $|\nu|:\mathcal{F}\to[0,\infty]$ is a positive measure.
Later we see that $|\nu|$ is a finite measure.
The total variation of ν is $\|\nu\|_1=|\nu|(\Omega)$.

<u>Continuity</u>: if ν is a complex measure on $\mathcal{F}$ & $(A_n)\subset\mathcal{F}$, then
(i) if $A_n\subset A_{n+1}$ $\forall n$, then $\nu(\bigcup_n A_n)=\lim_{n\to\infty}\nu(A_n)$
(ii) if $A_n\supset A_{n+1}$ $\forall n$, then $\nu(\bigcap_n A_n)=\lim_{n\to\infty}\nu(A_n)$

<u>Signed measure</u>: Ω set, $\mathcal{F}$ σ-field on Ω.

A signed measure on $\mathcal{F}$ is a countably additive set function $\nu:\mathcal{F}\to\mathbb{R}$.

Thm 2: Let Ω be a set, $\mathcal{F}$ σ-field on Ω, ν a signed measure on $\mathcal{F}$. Then $\exists$ measurable partition $\Omega=P\cup N$ of Ω s.t. $\forall A\in\mathcal{F}$, $A\subset P\Rightarrow\nu(A)\ge 0$, $\forall A\in\mathcal{F}$, $A\subset N\Rightarrow\nu(A)\le 0$.

<u>Rmks</u>: 1. The decomposition $\Omega=P\cup N$ is called the <u>Hahn-decomposition</u> of μ (or of Ω).
2. Let's define $\nu^+(A)=\nu(A\cap P)$, $\nu^-(A)=-\nu(A\cap N)$, $A\in\mathcal{F}$. Then ν^+,ν^- are finite positive measures such that $\nu=\nu^+-\nu^-$ & $|\nu|=\nu^++\nu^-$. These determine ν^+,ν^- uniquely and $\nu=\nu^+-\nu^-$ is the <u>Jordan decomposition</u> of ν.
3. If ν is a complex measure on $\mathcal{F}$, then $\operatorname{Re}(\nu)$, $\operatorname{Im}(\nu)$ and are signed measures with Jordan decompositions $\nu_1-\nu_2$, $\nu_3-\nu_4$, respectively. Then, $\nu=\nu_1-\nu_2+i\nu_3-i\nu_4$ — the Jordan decomposition of ν. Then $\nu_k\le|\nu|$, $1\le k\le 4$ & $|\nu|\le\nu_1+\nu_2+\nu_3+\nu_4$. So $|\nu|$ is a finite measure.
4. If ν is a signed measure on $\mathcal{F}$ with Jordan decomposition $\nu^+-\nu^-$, then $\nu^+(A)=\sup\{\nu(B)\mid B\in\mathcal{F}, B\subset A\}$, $A\in\mathcal{F}$.

<u>Pf of Thm 2</u>: Define $\nu^+(A)=\sup\{\nu(B)\mid B\in\mathcal{F}, B\subset A\}$ $\forall A\in\mathcal{F}$.
Then ν^+ is ≥ 0 & ν^+ is finitely additive.
<u>Key step</u> $\nu^+(\Omega)<\infty$.
by contradiction, assume not; construct sequences $(A_n),(B_n)$, $A_0=\Omega$.
$\nu^+(A_n)=\infty\to B_n\subset A_n$ $\nu(B_n)>n$ &
$A_{n+1}=B_n$ or $A_n\setminus B_n$ whichever has $\nu^+=\infty$. This will contradict σ-additivity.

To see this, note that $(A_n)_{n\in\mathbb{N}}$ is a decreasing family of events in $\mathcal{F}$. Then, by σ-additivity of ν, $\nu(\bigcap_n A_n)=\lim\nu(A_n)$. Thus, it can't be the case that "$A_n=B_n$" for $\nu(\bigcap A_n)<\infty$. Thus, $\exists N\in\mathbb{N}$ s.t. $\forall n\ge N$, $A_{n+1}=A_n\setminus B_n$.
This gives $\nu(\bigcap A_n)=\lim\nu(A_k)$, $k\ge N$
Now, $\nu(A_k)=\nu(A_k\setminus B_k)+\nu(B_k)>k>N$
$\Rightarrow\nu(A_{k+1})<-N+\nu(A_k)$
$\Rightarrow\nu(A_k)<\nu(A_{k-1})-k<\cdots<\nu(A_N)-k\downarrow-\infty$
violating the finiteness of $\nu(\bigcap_n A_n)$.

<u>$\exists P\in\mathcal{F}$ $\nu^+(\Omega)=\nu(P)$.</u> take (A_n), $\nu(A_n)>\nu^+(\Omega)-2^{-n}$
check $P=\bigcup_n\bigcap_{m\ge n}A_m$ works. Let $N=\Omega\setminus P$.
<u>Check:</u>
$\bigcap_{m\ge n}A_m$ is an increasing family of events $\nu(P)=\lim_{n\to\infty}\nu(\bigcap_{m\ge n}A_n)$.

Now, for $p\ge n$, consider $\bigcap_{n\le m\le p}A_n$:

we first see that: $\nu(\bigcap_{n\le m\le n+1}A)$ $\quad -2^{-n}-2^{-n-1}$
$= -\nu(A_n\cup A_{n+1})+\nu(A_n)+\nu(A_{n+1})$
$> -\nu^+(\Omega)+2\nu^+(\Omega)-2^{-n}-2^{-n-1} = \nu^+(\Omega)-2^{-n}-2^{-n-1}$
and by induction we see that:
$$\nu\Big(\bigcap_{n\le m\le n+p}A\Big)>\nu^+(\Omega)-\sum_{m=0}^{p}2^{-n-m}$$
$$\Rightarrow\nu\Big(\bigcap_{n\le m}A\Big)=\lim_{p\ge n}\nu\Big(\bigcap_{n\le m\le n+p}A_m\Big)\ge\nu^+(\Omega)-2^{-n+1}$$
$$\Rightarrow\nu(P)=\lim_{n\to\infty}\nu\Big(\bigcup_n\bigcap_{m\ge n}A_m\Big)\ge\nu^+(\Omega)$$
$\Rightarrow\nu(P)=\nu^+(\Omega)$. Now let $N=\Omega\setminus P$ and define $\bar\nu_+(E)=\nu(E\cap P)$ and $\bar\nu_-(E)=\nu(E\cap N)$. Note that $\nu(E\cap N)\le 0$ $\forall E\in\mathcal{F}$ for otherwise $\nu^+(\Omega)<\nu(E\cap N)+\nu(P)=\nu((E\cap N)\cup P)\le\nu^+(\Omega)$ ↯.
Thus, $\bar\nu_-$ is a negative measure.
We claim: $\nu(N)=\inf\{\nu(E)\mid E\in\mathcal{F}\}$.
Suppose otherwise, then $\exists E\in\mathcal{F}$, s.t.
$\nu(E)<\nu(N)\Rightarrow\nu(\Omega\setminus E)=\nu(\Omega)-\nu(E)>-\nu(N)+\nu(\Omega)$
$\Rightarrow\nu(\Omega\setminus E)>\nu(P)$ ↯.
Then, $\nu(N)=\inf\{\nu(E)\mid E\in\mathcal{F}\}$.
Now we can see that $\forall E\in\mathcal{F}$: $\nu(E\cap P)\ge 0$.
For suppose otherwise, i.e. $\exists E\in\mathcal{F}$ s.t. $\nu(E\cap P)<0\Rightarrow$
$\nu(N)\le\nu(N\cup(E\cap P))=\nu(N)+\nu(E\cap P)<\nu(N)$ ↯
Now, we show that $\forall E\in\mathcal{F}$:
$\nu(E\cap N)=\inf\{\nu(A)\mid A\subset E, A\in\mathcal{F}\}$.
Suppose not, then $\exists A\subset E$ s.t.
$\nu(A)<\nu(E\cap N)\Rightarrow\nu(A)=\nu(A\cap P)+\nu(A\cap N)<\nu(E\cap N)$
$\Rightarrow\nu(A\cap N)<\nu(E\cap N)$
$\Rightarrow\nu(E\cap N\setminus A\cap N)=\nu(E\cap N)-\nu(A\cap N)>0$ ↯
$(A\cap N\subset E\cap N)$
Finally, we observe that
$\nu(E\cap P)=\sup\{\nu(A)\mid A\subset E, A\in\mathcal{F}\}$
since if $\exists A\subset E$ s.t. $\nu(A)>\nu(E\cap P)$
$\nu(A\cap P)\ge\nu(A\cap P)+\nu(A\cap N)>\nu(E\cap P)$
$\Rightarrow\nu(E\cap P\setminus A\cap P)=\nu(E\cap P)-\nu(A\cap P)<0$ ↯
we have thus obtained the desired decomposition. □