Imperial College London

COURSEWORK 1

IMPERIAL COLLEGE LONDON

DEPARTMENT OF MATHEMATICS

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# MATH60030
# Fourier Analysis and the Theory of Distributions

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# Problems

## Question 1

### Part (a)

Let $f \in L^1(\mathbb{R})$. Show that

$$\sum_{j=-\infty}^{\infty} f(x + 2\pi j), \quad x \in \mathbb{R}$$

is in $L_1([-\pi, \pi])$.

### Part (b)

With $\phi$ as in part (a), let $c_n$ for $n$ in $\mathbb{N}$ be the Fourier series coefficients given by:

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(x) e^{-inx} dx$$

Show that the Fourier transform of $f$ evaluated at $n$

$$\mathcal{F}[f](n) = \int_{\mathbb{R}} f(x) e^{-inx} dx = c_n.$$

### Part (c)

Suppose that

$$\lim_{N \to \infty} 2\pi \sum_{n=-N}^{N} f(2\pi n) = \sum_{n=-\infty}^{\infty} c_n \tag{1}$$

Then deduce that

$$\sum_{n=-\infty}^{\infty} c_n = 2\pi \sum_{n=-\infty}^{\infty} f(2\pi n)$$

as required.

### Part (d)

For $t \neq 0$, set

$$f_t(x) = f(tx), \quad f \in L^1(-\infty, \infty)$$

yielding $f_t \in L^1(-\infty, \infty)$ for all $t$ as above. Show that

$$2\pi t \sum_{n=-\infty}^{\infty} f(2\pi t n) = \sum_{n=-\infty}^{\infty} \mathcal{F}[f]\left(\frac{n}{t}\right), \quad t \neq 0.$$

## Question 2

### Part (a)

Let

$$\phi_n(x) = H_n(x)\exp\left(-\frac{x^2}{2}\right), \quad x \in \mathbb{R}$$

and

$$H_n(x) = \begin{cases} 1, & n = 0 \\ (-1)^n \exp(x^2)\frac{d^n}{dx^n}\left[\exp(-x^2)\right], & n \geq 1 \end{cases}$$

Show that the sequence $(\phi_n)_{n\in\mathbb{N}}$ is an orthogonal one in $L^2(\mathbb{R})$.

### Part (b)

With $(\phi_n)_{n\in\mathbb{N}}$ and $(H_n)_{n\in\mathbb{N}}$ be as above. Show that

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \quad n \geq 1 \tag{2}$$

and

$$\frac{d}{dx}H_n(x) = 2xH_n(x) - H_{n+1}(x) = 2nH_{n-1}(x), \quad n \geq 1.$$

### Part (c)

Find eigenfunctions of the Fourier transform with eigenvalues $c_n = \sqrt{2\pi}(-i)^n$ for $n \geq 0$.

# 1 Solutions

## Question 1

### Part (a)

It suffices to show that

$$\phi_N(x) = \sum_{j=-N}^{N} f(x + 2\pi j), \quad x \in \mathbb{R}$$

is Cauchy in $L_1(-\pi,\pi)$. To this end, consider without loss of generality the norms for $n > m$ both in $\mathbb{N}$:

$$\|\phi_n - \phi_m\|_{L^1(-\pi,\pi)} = 2\pi\int_{[-\pi,\pi]}\left|\sum_{j=-n}^{n} f(x+2\pi j) - \sum_{j=-m}^{m} f(x+2\pi j)\right| dx \tag{3}$$

$$= 2\pi\int_{[-\pi,\pi]}\left|\sum_{j=m+1}^{n} f(x+2\pi j) + \sum_{j=-m-1}^{-n} f(x+2\pi j)\right| dx$$

$$\leq 2\pi \sum_{j=m+1}^{n} \int_{[-\pi,\pi]} |f(x+2\pi j)|dx + 2\pi \sum_{j=-m-1}^{-n} \int_{[-\pi,\pi]} |f(x+2\pi j)|dx$$

by the triangle inequality. Furthermore, by the change of variables formula for the Lebesgue integral, one obtains:

$$||\phi_n - \phi_m||_{L_1(-\pi,\pi)} \leq 2\pi \sum_{j=m+1}^{n} \int_{[-\pi+2\pi j,\pi+2\pi j]} |f(x)|dx + 2\pi \sum_{j=-m-1}^{-n} \int_{[-\pi+2\pi j,\pi+2\pi j]} |f(x)|dx$$

$$\leq 2\pi \int_{[2\pi m+\pi,\infty)} |f(x)|dx + 2\pi \int_{(-\infty,-2\pi m-\pi]} |f(x)|dx$$

since the sets

$$A_j = [-\pi + 2\pi j, \pi + 2\pi j], \quad j \in \mathbb{Z} \tag{4}$$

have disjoint interiors. Additionally, the fact that $f$ is in $L_1(-\infty,\infty)$ gives (by the Dominated convergence theorem) that

$$\int_{[M,\infty)} |f(x)|dx + \int_{(-\infty,M]} |f(x)|dx \to 0, \quad \text{as } M \to \infty$$

which yields that

$$\limsup_{n\geq m} ||\phi_n - \phi_m||_{L^11(-\pi,\pi)} \leq 2\pi \int_{[2\pi m+\pi,\infty)} |f(x)|dx + 2\pi \int_{(-\infty,-2\pi m-\pi]} |f(x)|dx \to 0, \quad m \to \infty$$

showing that the sequence of $\phi_N$ is Cauchy in $L_{[-\pi,\pi]}$, thereby converging to some $\phi$ in $L_{[-\pi,\pi]}$ by completeness, as required.

Now, by definition of the $\phi_N$ and the triangle inequality, we have for all $N \geq 1$:

$$||\phi_N||_{L^1[-\pi,\pi]} \leq 2\pi \sum_{j=-N}^{N} ||f(x+2\pi j)||_{L^1[-\pi,\pi]}$$

$$= 2\pi \sum_{j=-N}^{N} ||f(x)||_{L^1[-\pi+2\pi j,\pi+2\pi j]} = 2\pi \sum_{j=-N}^{N} ||f(x)\mathbb{1}_{A_j}||_{L^1(-\infty,\infty)} \leq 2\pi \cdot ||f||_{L^1(-\infty,\infty)}$$

using the fact that the sets (4) have pairwise disjoint interiors. Now, by the reverse triangle inequality, passing to the limit as $N \to \infty$ gives:

$$||\phi||_{L^1[-\pi,\pi]} \leq 2\pi \cdot ||f||_{L^1(-\infty,\infty)}$$

as required.

**Part (b)**

The Fourier series coefficients $c_n$ for $n$ in $\mathbb{N}$ are given by:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(x)e^{-inx}dx$$

Since $\phi_N \to \phi$ in $L^1_{[-\pi,\pi]}$,

$$\alpha_{n,N} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_N(x)e^{-inx}dx \to c_n, \quad N \to \infty$$

which we now compute. Now, by (3):

$$\begin{aligned}\alpha_{n,N} &= \sum_{j=-N}^{N}\int_{-\pi}^{\pi} f(x+2\pi j)e^{-inx}dx = \sum_{j=-N}^{N}\int_{-\pi+2\pi j}^{\pi+2\pi j} f(x)e^{-inx+2\pi ji}dx \\ &= \sum_{j=-N}^{N}\int_{-\pi+2\pi j}^{\pi+2\pi j} f(x)e^{-inx}dx = \int_{\mathbb{R}}\sum_{j=-N}^{N} \mathbb{1}_{A_j}(x)f(x)e^{-inx}dx\end{aligned}$$

with the $A_j$ as in (4). Since,

$$\left|\sum_{j=-N}^{N} \mathbb{1}_{A_j}(x)f(x)e^{-inx}\right| \le |f(x)| \in L^1_{(-\infty\infty)}$$

and

$$\sum_{j=-N}^{N} \mathbb{1}_{A_j}(x)f(x)e^{-inx} \to f(x)e^{-inx}, \quad N \to \infty$$

both almost everywhere, it follows from the Dominated convergence theorem that

$$\alpha_{n,N} = \int_{\mathbb{R}}\sum_{j=-N}^{N} \mathbb{1}_{A_j}(x)f(x)e^{-inx}dx \to \int_{\mathbb{R}} f(x)e^{-inx}dx = \mathcal{F}[f](n)$$

the Fourier transform of $f$ evaluated at $n$, as required.

**Part (c)**

The assumption in the statement of this question is equivalent to:

$$\lim_{N\to\infty} 2\pi \sum_{n=-N}^{N} f(2\pi n) = \sum_{n=-\infty}^{\infty} c_n \tag{5}$$

The previous sub-question then yields:

$$\sum_{n=-\infty}^{\infty} c_n = \sum_{n=-\infty}^{\infty} \mathcal{F}[f](n) = 2\pi\sum_{n=-\infty}^{\infty} f(2\pi n)$$

as required.

**Part (d)**

Now, for $t \neq 0$, one can define

$$f_t(x) = f(tx), \quad f \in L^1(-\infty, \infty)$$

yielding $f_t \in L^1(-\infty,\infty)$ for all $t$ as above. This means that one can replace $f$ with $f_t$ in questions one part (a) and (b) to deduce that

$$\sum_{n=-\infty}^{\infty} c_n^t = \sum_{n=-\infty}^{\infty} \mathcal{F}[f_t](n)$$

where

$$c_n^t = \frac{1}{2\pi}\int_{\pi}^{\pi} \phi_t(x)e^{-inx}dx$$

are the Fourier coefficients of

$$\phi_t(x) = L^1[-\pi,\pi] - \lim_{|N|\to\infty}\sum_{j=-N}^{N} f_t(x+2\pi j)$$

This means that the same assumptions as in question one part (c) for $\phi_t$ and $f_t$, enable one to arrive at:

$$\lim_{N\to\infty} 2\pi\sum_{n=-N}^{N} f_t(2\pi n) = 2\pi\sum_{n=-\infty}^{\infty} f(2\pi t n)$$

$$= \sum_{n=-\infty}^{\infty} c_n^t = \sum_{n=-\infty}^{\infty}\mathcal{F}[f_t](n) = \sum_{n=-\infty}^{\infty}\int_{\mathbb{R}} f_t(x)e^{-inx}dx$$

$$= \sum_{n=-\infty}^{\infty}\int_{\mathbb{R}} f(tx)e^{-inx}dx = \sum_{n=-\infty}^{\infty}\frac{1}{t}\int_{\mathbb{R}} f(x)e^{-i\frac{n}{t}x}dx$$

by the change of variables $tx \to x$, finally giving

$$2\pi t\sum_{n=-\infty}^{\infty} f(2\pi t n) = \sum_{n=-\infty}^{\infty}\int_{\mathbb{R}} f(x)e^{-i\frac{n}{t}x}dx = \sum_{n=-\infty}^{\infty}\mathcal{F}[f]\left(\frac{n}{t}\right), \quad t \neq 0$$

## Question 2

**Part (a)**

It suffices to check that the inner products:

$$(\phi_n,\phi_m)_{L^2(-\infty,\infty)} = \begin{cases} 0, & m \neq n \\ > 0, & m = n \end{cases}, \quad m,n \in \mathbb{N}$$

where

$$\phi_n(x) = H_n(x)\exp\left(-\frac{x^2}{2}\right), \quad x \in \mathbb{R}$$

and

$$H_n(x) = \begin{cases} 1, & n = 0 \\ (-1)^n \exp(x^2)\frac{d^n}{dx^n}\left[\exp(-x^2)\right], & n \geq 1 \end{cases}$$

We first show by induction on $n \geq 0$ that

$$\begin{cases} \frac{d^n}{dx^n}\left[\exp(-x^2)\right] = p_n(x)\left[\exp(-x^2)\right] \\ p_n(x) = (-1)^n \cdot 2^n x^n + \cdots \end{cases} \tag{6}$$

i.e., $p_n(x)$ is a polynomial of degree $n$ with leading coefficient $(-1)^n \cdot 2^n$. Now, assuming the inductive hypothesis (6), one can differentiate the first expression therein and apply the product rule to obtain:

$$\frac{d^{n+1}}{dx^{n+1}}\left[\exp(-x^2)\right] = \left[-2x \cdot p_n(x) + \frac{d}{dx}p_n(x)\right]\exp(-x^2)$$

Clearly,

$$p_{n+1}(x) := -2x \cdot p_n(x) + \frac{d}{dx}p_n(x)$$

is a polynomial of degree $n+1$ with leading coefficient $(-1)^{n+1} \cdot 2^{n+1}$. Noting that the case $n = 0$, holds trivially, we complete the proof by induction. This readily yields that

$$\begin{aligned} H_n(x) &= (-1)^n \exp(x^2) p_n(x)\left[\exp(-x^2)\right] \\ &= (-1)^n \left[(-1)^n \cdot 2^n x^n + \cdots\right] = 2^n x^n + \cdots, \quad n \geq 0. \end{aligned} \tag{7}$$

a polynomial of degree $n$ with leading coefficient $2^n$. Now, fix $n > m$ in $\mathbb{N}$ and compute:

$$\begin{aligned} (\phi_n, \phi_m)_{L^2(-\infty,\infty)} &= \int_{\mathbb{R}} \phi_n(x)\phi_m(x)dx \\ &= \int_{\mathbb{R}} \phi_n(x)\phi_m(x)dx = \int_{\mathbb{R}} H_n(x)\exp\left(-\frac{x^2}{2}\right)H_m(x)\exp\left(-\frac{x^2}{2}\right)dx \\ &= \int_{\mathbb{R}} H_n(x)H_m(x)\exp\left(-x^2\right)dx = (-1)^n \int_{\mathbb{R}} \frac{d^n}{dx^n}\left[\exp(-x^2)\right]H_m(x)dx \end{aligned}$$

Since

$$\frac{d^n}{dx^n}\left[\exp(-x^2)\right]H_m(x) = p_n(x)H_m(x)\exp(-x^2)$$

and

$$f(x)\exp(-x^2)$$

is integrable for all polynomials $f(x)$ and vanishes as $|x| \to \infty$, we can integrate by parts $m+1$ times to obtain (just note that $p_n(x)H_m(x)$ is polynomial in $x$):

$$(\phi_n, \phi_m)_{L^2(-\infty,\infty)} = (-1)^{n+m+1} \int_{\mathbb{R}} \frac{d^{n-m-1}}{dx^{n-m-1}}\left[\exp(-x^2)\right]\frac{d^{m+1}}{dx^{m+1}}H_m(x)dx = 0$$

since $H_m(x)$ is a polynomial of degree $m$. An identical computation also gives for $n \geq 0$:

$$(\phi_n, \phi_n)_{L^2(-\infty,\infty)} = (-1)^{2n} \int_{\mathbb{R}} \frac{d^{n-n}}{dx^{n-n}} \left[\exp(-x^2)\right] \frac{d^n}{dx^n} H_n(x) dx$$

$$= \int_{\mathbb{R}} \exp(-x^2) \frac{d^n}{dx^n} H_n(x) dx = 2^n \cdot n! \int_{\mathbb{R}} \exp(-x^2) dx = \sqrt{\pi} 2^n \cdot n! > 0$$

by (7) and the standard formula for the integral of a Gaussian. This shows that the system $\{\phi_n\}_{n\in\mathbb{N}}$ is indeed orthogonal.

**Part (b)**

For $n \geq 1$, consider $xH_n(x)$; it is an $n+1$ degree polynomial and hence can be expanded in terms of $H_k(x), k \leq n+1$, as they are linearly independent polynomials (this follows from the orthogonality of the $\phi_n(x)$). Thus. we have the expansion:

$$xH_n(x) = \alpha_{n+1} H_{n+1}(x) + \alpha_n H_n(x) + \alpha_{n-1} H_{n-1}(x) + \cdots$$

where the $\alpha$'s are real coefficients and can be computed as:

$$\alpha_k = \frac{(x\phi_n, \phi_k)_{L^2(-\infty,\infty)}}{(\phi_k, \phi_k)_{L^2(-\infty,\infty)}} = \frac{(\phi_n, x\phi_k)_{L^2(-\infty,\infty)}}{(\phi_k, \phi_k)_{L^2(-\infty,\infty)}}$$

by the orthogonality property of the $\phi_n$ again and the definition of the inner product as an integral. Now, notice that for $k \leq n-2$,

$$xH_k(x) \in \text{span}\{H_0, \cdots, H_{n-1}\}$$

thus, $(\phi_n, x\phi_k)_{L^2(-\infty,\infty)} = 0$ by orthogonality yielding $\alpha_k = 0$ and

$$xH_n(x) = \alpha_{n+1} H_{n+1}(x) + \alpha_n H_n(x) + \alpha_{n-1} H_{n-1}(x).$$

First, we notice that

$$\alpha_{n-1} = \frac{(x\phi_n, \phi_{n-1})_{L^2(-\infty,\infty)}}{(\phi_n, \phi_{n-1})_{L^2(-\infty,\infty)}} = \frac{\int_{\mathbb{R}} x \cdot \phi_n(x)\phi_{n-1}(x) dx}{(\phi_{n-1}, \phi_{n-1})_{L^2(-\infty,\infty)}}$$

$$= \frac{(-1)^n \int_{\mathbb{R}} xH_{n-1}(x) \frac{d^n}{dx^n} \exp(-x^2) dx}{(\phi_n, \phi_n)_{L^2(-\infty,\infty)}} = \frac{\int_{\mathbb{R}} \frac{d^n}{dx^n}[xH_{n-1}(x)] \exp(-x^2) dx}{(\phi_n, \phi_n)_{L^2(-\infty,\infty)}} = \frac{\sqrt{\pi} n! 2^{n-1}}{\sqrt{\pi}(n-1)! 2^{n-1}}$$

$$= n$$

Similarly, we have

$$\alpha_{n+1} = \frac{(x\phi_n, \phi_{n+1})_{L^2(-\infty,\infty)}}{(\phi_{n+1}, \phi_{n+1})_{L^2(-\infty,\infty)}} = \frac{\int_{\mathbb{R}} x \cdot \phi_n(x)\phi_{n+1}(x) dx}{(\phi_{n+1}, \phi_{n+1})_{L^2(-\infty,\infty)}}$$

$$= \frac{(-1)^{n+1} \int_{\mathbb{R}} xH_n(x) \frac{d^{n+1}}{dx^{n+1}} \exp(-x^2) dx}{(\phi_{n+1}, \phi_{n+1})_{L^2(-\infty,\infty)}} = \frac{\int_{\mathbb{R}} \frac{d^{n+1}}{dx^{n+1}}[xH_n(x)] \exp(-x^2) dx}{(\phi_{n+1}, \phi_{n+1})_{L^2(-\infty,\infty)}} = \frac{\sqrt{\pi}(n+1)! 2^n}{\sqrt{\pi}(n+1)! 2^{n+1}}$$

$$= \frac{1}{2}$$

Finally, notice that by the definition of the $H_n(x)$, they can only be either even or odd. This can be seen inductively. The base case is trivially true. For the inductive case, if $H_{n-1}(x)$ is either even or odd, one has that

$$H_{n-1}(x)\exp(-x^2)$$

is either even or odd. But, $H_n$ can be expressed as

$$H_n(x) = (-1)\exp(x^2)\frac{d}{dx}\left[H_{n-1}(x)\exp(-x^2)\right]$$

which is either even or odd, since derivatives of even functions are odd and vice versa. This enables us to show that

$$\alpha_n = \frac{(x\phi_n, \phi_n)_{L^2(-\infty,\infty)}}{(\phi_n, \phi_n)_{L^2(-\infty,\infty)}} = \frac{\int_{\mathbb{R}} x \cdot \phi_n(x)\phi_n(x)dx}{(\phi_n, \phi_n)_{L^2(-\infty,\infty)}}$$

$$= \frac{\int_{\mathbb{R}} xH_n^2(x)\exp(-x^2)dx}{(\phi_n, \phi_n)_{L^2(-\infty,\infty)}} = 0$$

since $xH_n^2(x)\exp(-x^2)$ is an odd integrable function. This finally yields the desired equality

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \quad n \geq 1 \tag{8}$$

Now, for the second property:

$$\frac{d}{dx}H_n(x) = \frac{d}{dx}\left[(-1)^n\exp(x^2)\frac{d^n}{dx^n}\exp(-x^2)\right]$$

$$= 2x(-1)^n\exp(x^2)\frac{d^n}{dx^n}\exp(-x^2) - (-1)^{n+1}\exp(x^2)\frac{d^{n+1}}{dx^{n+1}}\exp(-x^2)$$

$$= 2xH_n(x) - H_{n+1}(x) = 2nH_{n-1}(x), \quad n \geq 1 \tag{9}$$

by (8).

**Part (c)**

I claim that the $\phi_n(x)$ are eigenvectors of the Fourier transform with eigenvalues $c_n = \sqrt{2\pi}(-i)^n$ for $n \geq 0$. The base case $n = 0$ can be checked directly giving

$$\mathcal{F}[\phi_0(x)](\lambda) = \int_{\mathbb{R}} \exp\left(-\frac{x^2}{2}\right)\exp(-i\lambda x)dx$$

$$= \sqrt{2\pi}\exp\left(-\frac{\lambda^2}{2}\right) = c_0\phi_0(\lambda)$$

by example four on pages 35-36 of the lecture notes. Now for the inductive part, suppose that

$$\mathcal{F}[\phi_n(x)](\lambda) = c_n \phi_n(x)$$

and

$$\mathcal{F}[\phi_{n-1}(x)](\lambda) = c_{n-1} \phi_{n-1}(x)$$

with $c_n, c_{n-1}$ as above. One now computes

$$\mathcal{F}[\phi_{n+1}(x)](\lambda) = \int_{\mathbb{R}} H_{n+1}(x) \exp\left(-\frac{x^2}{2}\right) \exp(-i\lambda x) dx$$

$$= \int_{\mathbb{R}} (2xH_n(x) - 2nH_{n-1}(x)) \exp\left(-\frac{x^2}{2}\right) \exp(-i\lambda x) dx$$

$$= \int_{\mathbb{R}} 2xH_n(x) \exp\left(-\frac{x^2}{2}\right) \exp(-i\lambda x) dx - 2n\mathcal{F}[\phi_{n+1}(x)](\lambda)$$

using (8). Furthermore, integration by parts yields:

$$\int_{\mathbb{R}} 2xH_n(x) \exp\left(-\frac{x^2}{2}\right) \exp(-i\lambda x) dx$$

$$= -2\left[H_n(x) \exp\left(-\frac{x^2}{2}\right) \exp(-i\lambda x)\right]_{-\infty}^{\infty} + 2\int_{\mathbb{R}} \frac{d}{dx}(H_n(x) \exp(-i\lambda x)) \exp\left(-\frac{x^2}{2}\right) dx$$

$$= 2\int_{\mathbb{R}} \left(\frac{d}{dx} H_n(x) - i\lambda H_n(x)\right) \exp(-i\lambda x) \exp\left(-\frac{x^2}{2}\right) dx$$

Now, using property (9):

$$= 2\int_{\mathbb{R}} (2nH_{n-1}(x) - i\lambda H_n(x)) \exp(-i\lambda x) \exp\left(-\frac{x^2}{2}\right) dx$$

$$= 4n\mathcal{F}[\phi_{n-1}(x)](\lambda) - 2i\lambda\mathcal{F}[\phi_n(x)](\lambda)$$

Combining everything together, we have:

$$\mathcal{F}[\phi_{n+1}(x)](\lambda) = 4n\mathcal{F}[\phi_{n-1}(x)](\lambda) - 2i\lambda\mathcal{F}[\phi_n(x)](\lambda) - 2n\mathcal{F}[\phi_{n+1}(x)](\lambda)$$

$$= 2n\mathcal{F}[\phi_{n-1}(x)](\lambda) - 2i\lambda\mathcal{F}[\phi_n(x)](\lambda)$$

Finally, the induction hypotheses imply:

$$\mathcal{F}[\phi_{n+1}(x)](\lambda) = 2nc_{n-1}\phi_{n-1}(\lambda) - i\lambda c_n\phi_n(\lambda) = 2nc_{n-1}\phi_{n-1}(\lambda) + 2(-i)^2 c_{n-1}\lambda\phi_n(\lambda)$$

$$= -c_{n-1}\phi_{n+1}(\lambda) = (-i)^2 c_{n-1}\phi_{n+1}(\lambda) = c_{n+1}\phi_{n+1}(\lambda)$$

using property (8) and that $c_n = (-i)^n = (-i)c_{n-1}$. This completes the proof by induction as required.