Brownian Regularity of the KPZ fixed point

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December 18, 2025

Overview

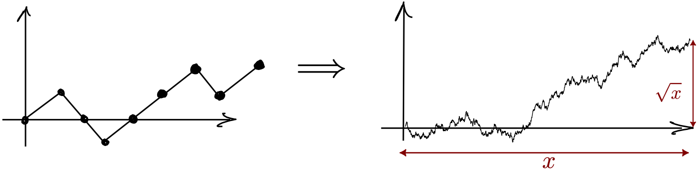
1. Gaussian universality
2. The KPZ fixed point and the universality class
3. "Brownianness" of the KPZ fixed point
4. Proofs

Gaussian universality: Brownian motion

Universality

Often, if some random outcome is a result of many different sources of randomness, the details of those randomness do not matter much.

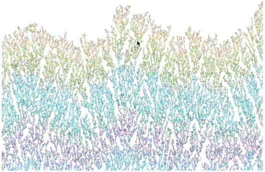
- Brownian motion was observed by [Brown, 1828] as movement of pollens in water.
- [Bachelier, 1900], [Einstein, 1905], [Smoluchowski, 1906], ...
- [Wiener, 1923] gave mathematical description of the sample paths.
- Universality: [Donsker, 1951] Brownian motion is the scaling limit of a random walk, or discrete-time stochastic processes with stationary independent increments (Fluctuation: Space $= 1 : 2$).



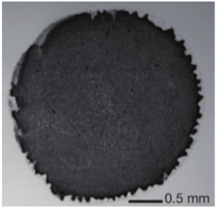
Beyond Gaussian universality

Many naturally occurring growth models exhibit the following features:

1. Height function change depends only on nearby heights
2. Independent space-time noise
3. Lateral growth depends non-linearly on the slope: Vertical effective growth rate depends nonlinearly on local slope.
4. Relaxation mechanism

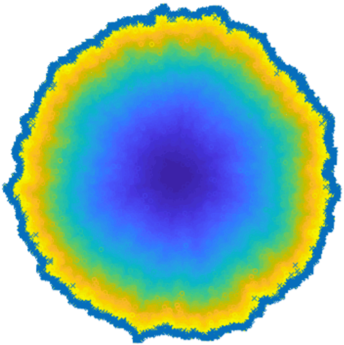


Tetris (randomized)



Coffee stains

Eden model



- Introduced by [Eden et al., 1961] to study biological growth clusters on a surface.
- Neighboring sites get added to the cluster at rate 1.

Fire front

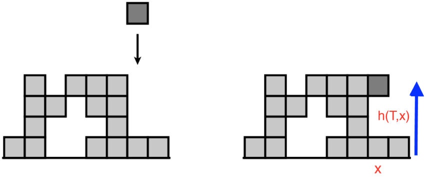
Ballistic deposition

- Ballistic deposition model: Introduced by [Vold, 1959] to model sedimentation of colloids

Ballistic deposition model

Unit blocks fall independently and in parallel from the sky above each site of $\mathbb{Z}$ at rate 1 and **stick to the first edge against which it becomes incident**.

- Fluctuation: $h(T,x) - c_1 T$ of order $T^{1/3}$.
- Transversal correlation in the scale of $T^{2/3}$.
- Scaling limit: $c_2 T^{-1/3}(h(T,0) - c_1 T)$ converges to *Tracy-Widom* distribution from random matrix theory.



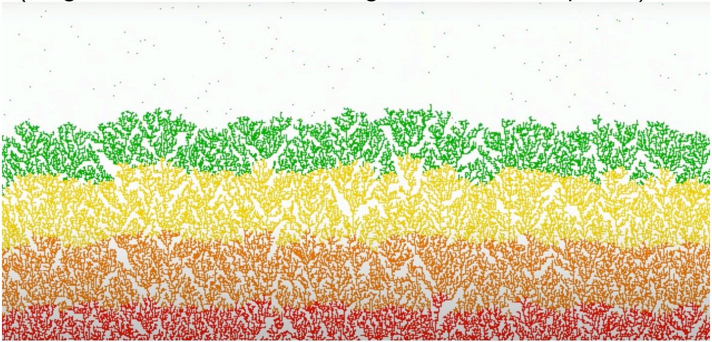
Ballistic deposition run for long time

- Full scaling limit: $c_2 T^{-1/3}(h(T, T^{2/3}x) - c_1 T)$, as a process in x, is conjectured to converge as $T \to \infty$ to a universal process.
- Fluctuation : Space : Time $= 1:2:3$

$$h_\varepsilon(t,x) := \varepsilon^{1/2} h(\varepsilon^{-3/2}t, \varepsilon^{-1}x) - C_\varepsilon t$$

is conjectured to converge as $\varepsilon \to 0$ to a universal process $h(\cdot,\cdot)$.

(Plug in $t = 1$ and $T = \varepsilon^{-3/2}$ to get back the first equation)



Overview

The Kardar-Parisi-Zhang equation (1986)

VOLUME 56, NUMBER 9 PHYSICAL REVIEW LETTERS 3 MARCH 1986

Dynamic Scaling of Growing Interfaces

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(Received 12 November 1985)

A model is proposed for the evolution of the profile of a growing interface. The deterministic growth is solved exactly, and exhibits nontrivial relaxation patterns. The stochastic version is studied by dynamic renormalization-group techniques and by mappings to Burgers's equation and to a random directed-polymer problem. The exact dynamic scaling form obtained for a one-dimensional interface is in excellent agreement with previous numerical simulations. Predictions are made for more dimensions.

PACS numbers: 05.70.Ln, 64.60.Ht, 68.35.Fx, 81.15.Jj

$$\partial_t h = \underbrace{(\partial_x h)^2}_{\text{lateral growth}} + \underbrace{\partial_x^2 h}_{\text{relaxation}} + \underbrace{\xi}_{\substack{\text{space-time}\\ \text{white noise}}} \qquad \underbrace{h(t,x)}_{\substack{\text{height at time}\\ \text{t position x}}}$$

What is the KPZ universality class?

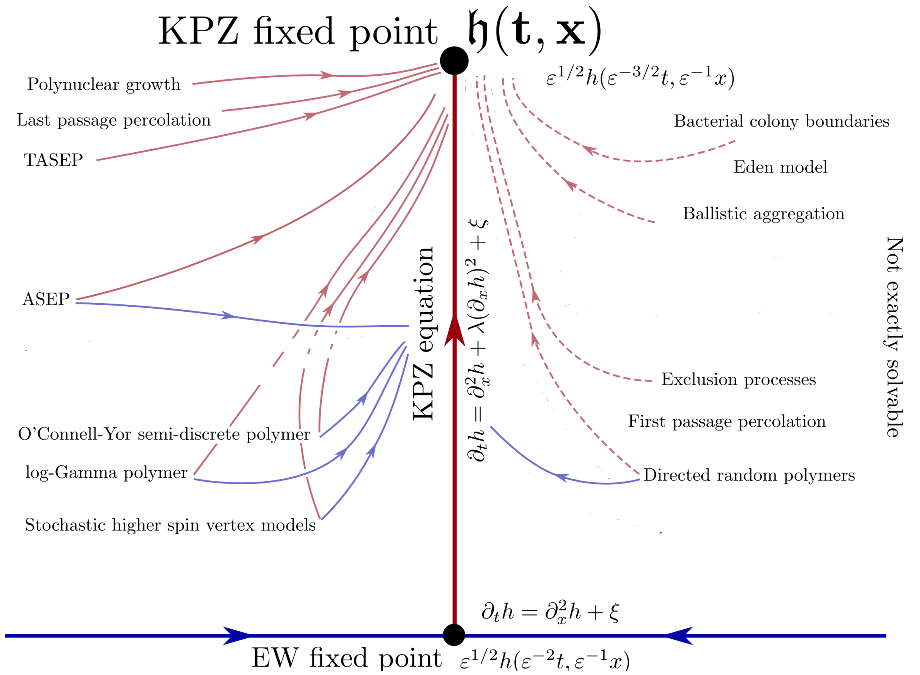
- All models in this class have a height function which has the "same asymptotic fluctuations" under the 1:2:3 scaling as the KPZ equation.

More precisely, the height function started from h_0 is to converge under the $1:2:3$ scaling to a universal object $h_t(\cdot; h_0)$ that

- is a Markov process in t
- whose distribution is fixed under $1:2:3$ scaling, i.e., for all $\varepsilon > 0$, $\varepsilon^{1/2} h(\varepsilon^{-3/2} t, \varepsilon^{-1} x; \varepsilon^{-1/2} h_0(\varepsilon x)) \overset{dist.}{=} h(t, x; h_0)$.

This universal object is the KPZ fixed point [Matetski et al., 2021]

- Analogue of the Donsker's theorem in this universality class!
- *A model is in the KPZ universality class, if starting from* **any height**, *its height under* $1:2:3$ *scaling converges to the KPZ fixed point*.


KPZ fixed point $\mathfrak{h}(\mathbf{t},\mathbf{x})$
$\varepsilon^{1/2}h(\varepsilon^{-3/2}t,\varepsilon^{-1}x)$
Polynuclear growth
Last passage percolation
TASEP
ASEP
O'Connell-Yor semi-discrete polymer
log-Gamma polymer
Stochastic higher spin vertex models
KPZ equation
$\partial_t h = \partial_x^2 h + \lambda(\partial_x h)^2 + \xi$
Bacterial colony boundaries
Eden model
Ballistic aggregation
Exclusion processes
First passage percolation
Directed random polymers
Not exactly solvable
$\partial_t h = \partial_x^2 h + \xi$
EW fixed point $\varepsilon^{1/2}h(\varepsilon^{-2}t,\varepsilon^{-1}x)$

KPZ fixed point for special initial conditions

- For the narrow wedge initial condition, $h_0(0) = 0$ and $h_0(x) = -\infty$ elsewhere, $h_1(\cdot)$ is the (parabolically shifted) Airy$_2$ process.
- For $h_0 \equiv 0$, the flat initial condition, $h_1(\cdot)$ is called the Airy$_1$ process.
- Brownian motion is invariant (except for height shift) for the KPZ fixed point.
- That is, the KPZ fixed point started from a Brownian initial condition is again a (different) Brownian motion.
- What can we say about the KPZ fixed point started from arbitrary initial condition?
- Does it still look "locally like a Brownian motion"?

Overview

Definitions: local Brownianness

Local Brownian limit

F has local Brownian limit if the scaled process $\varepsilon^{-1/2}(F(\cdot + \varepsilon\cdot') - F(\cdot)) \xrightarrow{d} B(\cdot')$ as $\varepsilon \to 0$, where B is a Brownian motion.

Brownian on compact

F is called Brownian on compact if for all $y_1 < y_2$, the law of $F(\cdot) - F(y_1)$ on $[y_1, y_2]$ is absolutely continuous wrt that of a Brownian motion (with diffusion parameter 2) starting from $(y_1, 0)$ on $[y_1, y_2]$.

- Brownian on compact $\implies$ local Brownian limit.

Long quest

A central question in the study of the KPZ universality class was the local Brownian nature of the KPZ fixed point.

- [Corwin and Hammond, 2014]: Brownian on compacts for Airy_2
- [Calvert et al., 2019]: Bound on RN derivative for Airy_2
- [Calvert et al., 2019]: For general initial condition– Brownian patchwork quilt. Question left open: if it can be reduced to a single patch?
- [Matetski et al., 2021]: Hölder $1/2-$ continuity and local limit convergence for KPZ fixed point
- [PR Pimentel, 2018, Pimentel, 2020]: local Brownian limit and Hölder continuity for certain initial conditions
- [Dauvergne, 2024]: L^∞ bound on Radon-Nikodym derivative of Airy_2 (up to height shift) on compacts.

KPZ fixed point is locally Brownian

Theorem, [Sarkar and Virág, 2021]

For all finitary initial conditions h_0, the KPZ fixed point at time t is Brownian on compacts.

Application:

- KPZ fixed point has a unique maxima on compacts (Johansson's conjecture!). In particular, Airy_1 process has a unique maxima on compacts.

Quantitative Brownian regularity

- $\mu \ll \nu$ means for any $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $\nu(A) < \delta$ one has $\mu(A) < \varepsilon$.
- Not very useful unless we have a relation between ε and δ.
- Such a relation can be obtained by the growth condition on the Radon-Nikodym derivative $f = \frac{d\mu}{d\nu}$.
- In particular, if $f \in L^p$ for some $p > 1$, then by Hölder's inequality

$$\mu(A) \leq C(\nu(A))^{1-\frac{1}{p}}.$$

- Thus bounds on the RN derivative of the KPZ fixed point wrt Brownian motion will bound the KPZ fixed point probabilities by Brownian probabilities, which are easy to compute!!

Conjecture, [Calvert et al., 2019]

The Radon-Nikodym derivative of the KPZ fixed point with arbitrary initial data with respect to Brownian motion is in L^p for all $p > 1$!

Main result

Quantitative Brownian regulrity, [Tassopoulos and Sarkar, 2025a]

The law of the KPZ fixed point started from arbitrary (finitary) initial data on some fixed interval, μ, exhibits a form of quantitative Brownian regularity of the form

$$\mu(A) \leq c \exp\left(-d \log^r \log\left(1/\nu(A)\right)\right) ,$$

for some positive constants $c, d > 0, r \in (0,1)$ and all events A, where ν denotes the Brownian measure.

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KPZ fixed point geometry: The Airy line ensemble and the Airy Sheet

- The proof uses a geometric representation of the KPZ fixed point as a variational formula.

- The parabolically shifted Airy line ensemble is a sequence of random continuous functions $\mathcal{A}_1 > \mathcal{A}_2 > \ldots$, introduced by [Prähofer and Spohn, 2002] in the version $\mathcal{A}_i(t) + t^2$, which is stationary in time.

- The top line $\mathcal{A}_1$ is the (parabolically shifted) Airy$_2$ process that appear as the limiting spatial fluctuation of random growth models starting from a single point.

- The Airy sheet was constructed by [Dauvergne et al., 2022]. The standard Airy sheet $\mathcal{S} : \mathbb{R}^2 \mapsto \mathbb{R}$ is a random continuous function defined as a last passage percolation across the Airy line ensemble.

Airy sheet profile

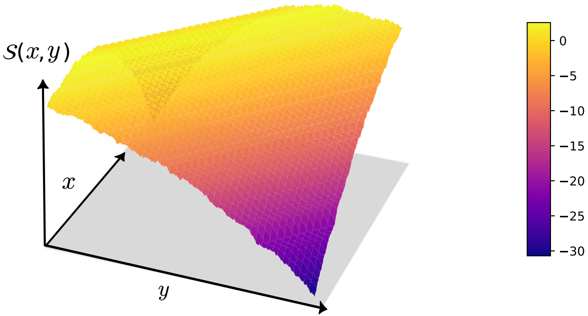
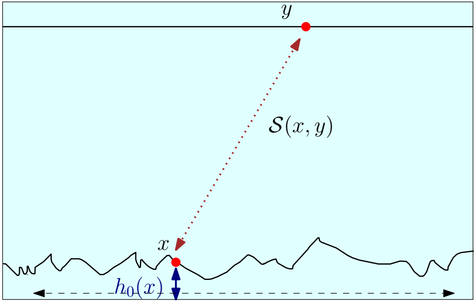


Figure: Simulation of the Airy sheet on $[-3, 3] \times [-3, 3]$ using approximation by Brownian last passage values. Note the parabolic curvature.

KPZ fixed point as a variational formula

- The KPZ fixed point at time 1 can be written in terms of the Airy sheet $\mathcal{S}$ and the initial condition $h_0 : \mathbb{R} \mapsto \mathbb{R} \cup \{-\infty\}$ as

$$h_1(y) = \sup_{x \in \mathbb{R}} \Big(h_0(x) + \mathcal{S}(x, y) \Big), \qquad y \in \mathbb{R}.$$



Fixed point reduction

- We proceed via a 'localisation argument': parabolic curvature of Airy profile allows us to restrict the variational formula to a compact set K (depending on h_0 and the compact set where we want to evaluate the KPZ fixed point).
- Using the variational formula of the KPZ fixed point and the Airy LPP representation of the Airy sheet, the KPZ fixed point at time 1 can be written as:

$$h_1(y) = \max_{1 \leq \ell \leq L_0} \left(G_\ell^{h_0} + \mathcal{A}[(0,\ell) \to (y,1)] \right).$$

- Geodesic intercept
- 'Boundary data' $G_\ell^{h_0}$
- Last passage values over Airy line ensemble

Geodesic transversal fluctuations

We obtain 'stretched exponential' concentration of geodesic intercepts L_0,

Theorem, [Tassopoulos and Sarkar, 2025a]

$$\sup_{k\in\mathbb{N}} \exp(dk^{1/126}) \cdot \mathbb{P}(L_0 \geq k) < \infty \,,$$

for some $d > 0$ and can also keep track of dependence on end point and 'asymptotic slope' of the geodesic.

- This was achieved by establishing quantitative control over jump times of last passage geodesics in the prelimiting (Brownian melon) environments converging to the Airy line ensemble (under edge-scaling).

Transversal fluctuations of semi-infinite geodesics

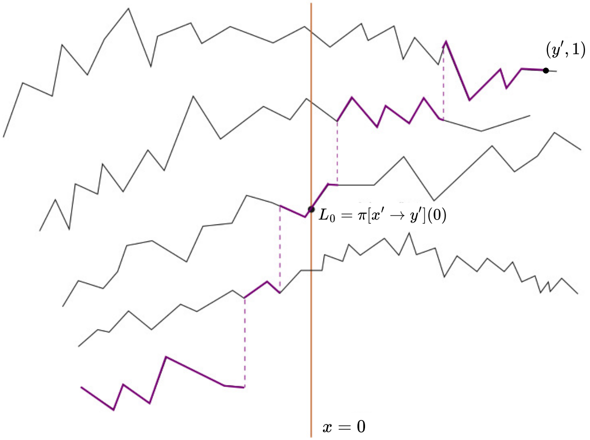


Figure: Above is displayed the point $(0, L_0)$ at which the last passage path $\pi[x' \to y']$ on the Airy line ensemble $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \cdots)$ (purple) meets with the axis $\{x = 0\}$, where $x', y' > 0$. Here $L_0 = 3$ and the first four lines of $\mathcal{A}$ are shown.

Boundary data

Control on boundary data $G_\ell^{h_0}$ is obtained through controlling geodesic coalescence on the Airy line ensemble.

- Boundary data $G_\ell^{h_0}$, $\ell \geq 1$ can be estimated for some $x_0 > 0$ in the max-plus support of initial data h_0, K by
  - pointwise bounds on the initial data on K
  - pointwise bounds on $G_1^{h_0}$ which is coupled to the Airy sheet
  - $\lim_{k\to\infty} |\mathcal{A}[(-\sqrt{k/(2x)}, k) \to (0,\ell)] - \mathcal{A}[(-\sqrt{k/(2x)}, k) \to (0,1)]|$, $x > 0$.
- Theorem 3.7 in [Sarkar and Virág, 2021] shows the limit a.s. stabilises eventually for $k \geq K_x$, (due to the parabolic 'curvature' of the Airy line ensemble). In [Tassopoulos and Sarkar, 2025a], we obtain explicit tails for such 'coalescence' depths K_x, keeping track of dependence on ℓ. In particular, we have for any $\theta > 0$,

$$\inf\left\{m \geq 1 : \mathbb{P}(K_x \geq m) \leq \theta\right\} \leq C\ell^{256}\left(\exp\left(d\log^{1000}(1/\theta)\right)\right)$$

for some $C, d > 0$.

Airy line ensemble last passage values

To obtain control over $\mathcal{A}[(0,\ell) \to (y,1)]$, $y \geq 0, \ell \geq 1$ we use the **Brownian Gibbs resampling property** of the Airy Line ensemble [Corwin and Hammond, 2014], which briefly states that for $a < b$, $k \in \mathbb{N}$, the law of the Airy line ensemble restricted to the region

$$\{1, 2, \cdots, k\} \times (a, b)\,,$$

$\mathcal{A}|_{\{1,2,\cdots,k\}\times(a,b)}$, conditionally on all the data generated by the Airy line ensemble outside of this region,

$$\{A_i(x) : (i,x) \notin [\![1,k]\!] \times (a,b)\}\,,$$

is given by k independent Brownian bridges with entry data $\underline{x} = (\mathcal{A}_i(a))_{1\leq i\leq k}$, $\underline{y} = (\mathcal{A}_i(b))_{1\leq i\leq k}$ conditioned to not intersect and stay above $f = \mathcal{A}_{k+1}$ on (a,b).

Brownian Gibbs

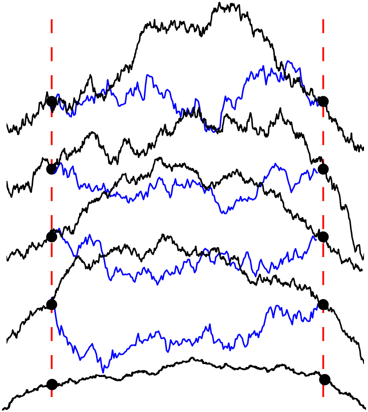


Figure: Figure illustrating the Brownian Gibbs property on the first four lines of the parabolic Airy Line ensemble $\mathcal{A} = \{\mathcal{A}_1 > \mathcal{A}_2 > \dots\}$ (in **black**) between two points (indicated by the red vertical dashed lines). The blue curves represent resampled versions of first four lines in the ensemble between the endpoints, conditioning on everything else and avoiding the fifth line.

Reduction to inhomogeneous Brownian LPP

The above bounds allow us with high probability to reduce the problem of Brownian regularity of the KPZ fixed point to that of the **inhomogeneous Brownian last passage values**,

$$\max_{k\leq\ell\leq m}\left(g_\ell + B[(0,\ell)\to(t,k)]\right), \quad t\geq 0$$

for $1\leq k\leq m$ and $g_1\geq\ldots\geq g_m$. In [Sarkar and Virág, 2021], it was already demonstrated that its top line is absolutely continuous with respect to Brownian motion on compacts. We strengthened this comparison and obtained the optimal estimates.

Radon-Nikodym derivarive estimates for Inhom. BLPP, [Tassopoulos and Sarkar, 2025b]

The Radon Nikodym derivative of the law of $\max_{1\leq\ell\leq m}(g_\ell + B[(0,\ell)\to(t,1)])$ away from zero against a rate two Brownian motion on compacts is in $L^{\infty-}$. In particular, the L^p norms, for $p>1$, grow like $\mathscr{O}_p(\mathrm{e}^{d_p m^2\log m})$, for some $d_p>0$.

Inhomogeneous Brownian LPP

The estimates on the Radon-Nikodym derivatives of inhomogeneous BLPP are obtained observing that

- the inhomogeneous BLPP values can be represented as a **finite** number of **iterated Pitman transforms** of Brownian motions;
- this gives a semi-martingale decomposition of the inhomogeneous BLPP values in terms of Brownian motions with singular drift given by local time terms;
- the above process can then be seen as a regular conditional distribution of the 'diagonal section' of Warren's Brownian motion, [Warren, 2007];
- this allows us to use explicit transition densities of the latter process to obtain bounds on the Radon-Nikodym derivatives, by relating the inhomogeneous case to the homogeneous one
- and use the fact that the top line of homogeneous BLPP is actually Dyson Brownian motion (through a continuous analogue of the RSK correspondence, [O'Connell and Yor, 2002]).

Future directions!

- Can we prove the Radon-Nikodym derivative is in L^p for all $p > 1$?

- Can we distinguish the KPZ fixed points started from different initial data?

- Ευχαριστώ για την προσοχή σας!

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