Imperial College London

Coursework 2

Imperial College London

Department of Mathematics

MATH70135
Analytic Methods In PDE

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Date: December 4, 2022

Problems

1. Let $u : \mathbb{R}^n \to R$ be continuous, non-negative and bounded. Prove using Harnack's inequality that u is a constant.

2. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $g \in C^0(\partial\Omega)$ and $f \in C^0(\overline{\Omega})$ such that $\Delta u = f$ in Ω.

   Show that
$$\max_{\overline{\Omega}} |u| \leq \max_{\partial\Omega} |g| + CM = C\left(\max_{\partial\Omega} |g| + \sup_{\overline{\Omega}} |f|\right)$$
   where C is a positive constant, that only depends on the domain $\overline{\Omega}$ and not on u, f, g.

3. Let $H_0^2(\Omega)$ denote the closure of the space $C_0^\infty(\Omega)$ with the H^2 norm; it is a closed subspace of the Hilbert space H^2, thus a Hilbert space in its own right. Consider the map,
$$B : H_0^2(\Omega) \times H_0^2(\Omega) \to \mathbb{R}$$
   where
$$B[u,v] = \int_\Omega \Delta u \Delta v, \quad u, v \in H_0^2(\Omega) \tag{1}$$
   Show that for any bounded linear functional f on $H_0^2(\Omega)$, there exists a unique $u \in H_0^2(\Omega)$ such that
$$B[u,v] = f[v], \quad \forall v \in H_0^2(\Omega)$$

4. [1, p.107] For $p \neq 2$ and $t \neq 0$ show that
$$\sup_{\varphi \in \mathcal{S} \setminus 0} \frac{\|S(t)\varphi\|_{L^p}}{\|\varphi\|_{L^p}} = \infty$$

   Hint. Consider $\varphi = e^{-(a+ib)|x|^2/2}$ with $a > 0$ and $b \in \mathbb{R}$. Let a tend to zero or infinity depending on the value of p. Discussion. There is a general principle. Suppose that S is an operator which respects the L^r norm in the sense that
$$c\|\varphi\|_{L^r} \leq \|S\varphi\|_{L^r} \leq C\|\varphi\|_{L^r}$$

   If there are functions such that φ_n and $S\varphi_n$ are spread regularly over their support and the supp$(S\varphi_n)$ is incomparably smaller than supp(φ_n), then S cannot be bounded in L^p for any $p > r$. The reason is that if M_n(resp. m_n) is a typical magnitude for φ_n (resp. $S\varphi_n$), then respect for L^r yields $M_n^r \operatorname{vol}(\operatorname{supp}\varphi_n) \sim m_n^r \operatorname{vol}(\operatorname{supp} S\varphi_n)$. Thus $m_n/M_n \to \infty$, so S is not bounded on L^∞. Similarly, $\|S\varphi_n\|_{L^p}/\|\varphi_n\|_{L^p} \to \infty$ for any $p > r$.

   If there are functions whose support is compressed, one finds that S is not bounded on L^p for any $p < r$. In our case $r = 2$, the L^2 norm is conserved and Gaussians of size $a^{-1/2}$ are spread by $S(t)$ over a region of size $a^{1/2}t$ which, letting a tend to infinity (resp. 0), is incomparably larger (resp. smaller) than the original spread. The conclusion is that S is unbounded on L^p for all $p \neq 2$.

The test functions of the hint are suggested by the solutions

$$S(t)\delta = (4\pi i t)^{-d/2} e^{i|x|^2/4t}$$

and

$$S(t)\Big((-4\pi i t)^{-d/2} e^{-i|x|^2/4t}\Big) = \delta,$$

which spread from a point to all of $\mathbb{P}^d$ and contract from all of $\mathbb{R}^d$ to a point. These are extreme examples of dispersion.

The operator $S(t)$ is the Fourier mutiplier $\mathscr{F}^* e^{-it|\xi|^2} \mathscr{F}$ and the multiplier $e^{-it|\xi|^2}$ is smooth and bounded but gives an operator which is unbounded on L^p for $p \neq 2$. This discontinuity is not obvious. Viewed from the point of view of the multiplier, the problem comes from the fact that $e^{-it|\xi|^2}$ oscillates faster and faster as $|\xi|$ tends to infinity.

5. Let $u : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$ be such that $u(t, \cdot), u_t(t, \cdot) \in \mathcal{S}(\mathbb{R}^d)$ for all $t \in \mathbb{R}$. Suppose u satisfies the Schrödinger equation

$$i\partial_t u + \Delta u = u|u|^2. \tag{2}$$

Show that the energy

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^d} \nabla u^T \overline{\nabla u} + \frac{1}{4}\int_{\mathbb{R}^d} u^2 \overline{u}^2$$

is constant.

Solutions

1. From problem sheet five, Harnack's inequality was derived. Restating it one has for $f : \mathbb{R}^n \to \mathbb{R}, u \geq 0$ continuous on the ball $B(x_0, R) \subset \mathbb{R}^n$ and harmonic in its interior, one has for all $|x - x_0| = r < R$:

$$\frac{1 - \frac{r}{R}}{[1 + \frac{r}{R}]^{n-1}} f(x_0) \leq f(x) \leq \frac{1 + \frac{r}{R}}{[1 - \frac{r}{R}]^{n-1}} f(x_0) \tag{3}$$

   Since u is bounded on $\mathbb{R}^n$, take

$$M = \sup_{\mathbb{R}^n} |u| < \infty$$

   and define

$$\tilde{u} = u + M$$

   Now one has that $\tilde{u} \geq 0$ on $\mathbb{R}^n$ and that it is harmonic on $\mathbb{R}^n$. Fix $y \in \mathbb{R}^n$ arbitrary. For all $R > |y| + 1$, the conditions for Harnack's inequality (3) are met with $y \in B(0, |y| + 1) \subset B(0, R)$ yielding

$$\frac{1 - \frac{|y|+1}{R}}{[1 + \frac{y|+1}{R}]^{n-1}} \tilde{u}(0) \leq \tilde{u}(y) \leq \frac{1 + \frac{y|+1}{R}}{[1 - \frac{y|+1}{R}]^{n-1}} \tilde{u}(0)$$

   Taking $R \to \infty$ gives that

$$\frac{1 - \frac{|y|+1}{R}}{[1 + \frac{y|+1}{R}]^{n-1}} \quad \text{and} \quad \frac{1 + \frac{y|+1}{R}}{[1 - \frac{y|+1}{R}]^{n-1}} \quad \text{both } \to 1$$

   Finally giving that $\tilde{u}(0) \leq \tilde{u}(y) \leq \tilde{u}(0)$. Thus, $\tilde{u} \equiv \tilde{u}(0)$ giving that u is constant on $\mathbb{R}^n$ as required.

2. Assuming f is bounded on $\overline{\Omega} \subset \mathbb{R}^n$ (otherwise there is nothing to show), let $M = \sup_{x \in \overline{\Omega}} |f(x)|$. Consider the functions

$$\tilde{u}_+(x) = u(x) + M\frac{|x|^2}{2n}, \quad \tilde{u}_-(x) = u(x) - M\frac{|x|^2}{2n}, \quad x \in \overline{\Omega}$$

   and let $C = \sup_{x \in \overline{\Omega}} \frac{|x|^2}{2n} < \infty$. Note that they are both $C^2(\Omega) \cap C(\overline{\Omega})$, where $|.| = \|.\|_2$. By linearity, one has that

$$\Delta\tilde{u}_+ = \Delta u + M = f + M \geq 0, \quad \Delta\tilde{u}_- = \Delta u - M = f - M \leq 0 \quad \text{on} \quad \Omega.$$

   Thus, $\tilde{u}_+$ is sub-harmonic in Ω and sufficiently regular for the weak maximum principle to yield

$$\max_{\overline{\Omega}} \left(u + M\frac{|x|^2}{2n} \right) = \max_{\overline{\Omega}} \tilde{u}_+ = \max_{\partial\Omega} \tilde{u}_+ = \max_{\partial\Omega} \left(u + M\frac{|x|^2}{2n} \right) = \max_{\partial\Omega} \left(g + M\frac{|x|^2}{2n} \right)$$

Now, one has

$$u(x) \le \tilde{u}_+(x) = u(x) + M\frac{|x|^2}{2n} \le \max_{\overline{\Omega}}\left(u + M\frac{|x|^2}{2n}\right) = \max_{\partial\Omega}\left(g + M\frac{|x|^2}{2n}\right)$$

$$\le \max_{\partial\Omega} g + CM \le \max_{\partial\Omega}|g| + CM$$

by the sub-additivity of max. Thus, taking maxima over the compact $\overline{\Omega}$

$$\max_{\overline{\Omega}} u \le \max_{\partial\Omega}|g| + CM \tag{4}$$

Similarly, $-\tilde{u}_-$ is sub-harmonic in Ω and sufficiently regular for the weak maximum principle to yield

$$\max_{\overline{\Omega}}\left(-u + M\frac{|x|^2}{2n}\right) = \max_{\overline{\Omega}} -\tilde{u}_- = \max_{\partial\Omega} -\tilde{u}_- = \max_{\partial\Omega}\left(-u + M\frac{|x|^2}{2n}\right) = \max_{\partial\Omega}\left(-g + M\frac{|x|^2}{2n}\right)$$

and by the same reasoning, one obtains

$$-u(x) \le \tilde{u}_-(x) = -u(x) + M\frac{|x|^2}{2n} \le \max_{\overline{\Omega}}\left(-u + M\frac{|x|^2}{2n}\right) = \max_{\partial\Omega}\left(-g + M\frac{|x|^2}{2n}\right)$$

$$\le \max_{\partial\Omega} -g + CM \le \max_{\partial\Omega}|g| + CM$$

yielding

$$\max_{\overline{\Omega}} -u \le \max_{\partial\Omega}|g| + CM \tag{5}$$

Since $|u| = \max\{u^+, u^-\}$, where $u^+ = \max\{u, 0\}, u^- = \max\{-u, 0\}$, the inequalities 4 and 5 imply for $\tilde{C} = \max\{1, C\}$:

$$\max_{\overline{\Omega}}|u| \le \max_{\partial\Omega}|g| + CM = \tilde{C}(\max_{\partial\Omega}|g| + M) = \tilde{C}\left(\max_{\partial\Omega}|g| + \sup_{\overline{\Omega}}|f|\right)$$

as required. Finally, note that $\tilde{C} = \max\left\{1, \sup_{x\in\overline{\Omega}}\frac{|x|^2}{2n}\right\}$, only depends on the domain $\overline{\Omega}$ and not on u, f, g.

3. $H_0^2(\Omega)$, defined as the closure of the space $C_0^\infty(\Omega)$ with the H^2 norm is a closed subspace of the Hilbert space H^2, thus a Hilbert space in its own right. Consider the map,

$$B : H_0^2(\Omega) \times H_0^2(\Omega) \to \mathbb{R}$$

where

$$B[u,v] = \int_\Omega \Delta u \Delta v, \quad u, v \in H_0^2(\Omega) \tag{6}$$

By the linearity of the Lebesgue integral, clearly the map B is bilinear. If one maages to show that this map B is both bounded and coercive, then Lax-Milgram from lectures can be used to obtain that for any bounded linear functional f on $H_0^2(\Omega)$, there exists a unique $u \in H_0^2(\Omega)$ such that

$$B[u,v] = f[v], \quad \forall v \in H_0^2(\Omega)$$

Now, boundedness follows from

$$|B[u,v]|^2 = \left| \int_\Omega \Delta u \Delta v \right|^2 \leq \left(\int_\Omega |\Delta u| \cdot |\Delta v| \right)^2 \leq \int_\Omega |\Delta u|^2 \cdot \int_\Omega |\Delta u|^2$$

by Cauchy-Schwarz. Notice also for $u \in H_0^2(\Omega)$ that

$$\int_\Omega |\Delta u|^2 = \int_\Omega \left(\sum_{i=1}^n \partial_i^2 u \right)^2 = \int_\Omega \left(\sum_{i=1}^n 1 \cdot \partial_i^2 u \right)^2 = n \cdot \int_\Omega \sum_{i=1}^n \left(\partial_i^2 u \right)^2 \leq n \cdot \|u\|_{H_0^2(\Omega)}^2$$

by another application of Cauchy Schwarz to the integrand. Thus, by a two-fold application of the above:

$$|B[u,v]|^2 \leq \int_\Omega |\Delta u|^2 \cdot \int_\Omega |\Delta u|^2 \leq n^2 \cdot \|u\|_{H_0^2(\Omega)}^2 \cdot \|v\|_{H_0^2(\Omega)}^2$$

or equivalently,

$$|B[u,v]| \leq \|u\|_{H_0^2(\Omega)} \cdot \|v\|_{H_0^2(\Omega)} \tag{7}$$

Now, for coercivity, using (6)

$$B[u,u] = \int_\Omega (\Delta u)^2 \tag{8}$$

Since the domain Ω is bounded, and $u \in H_0^2(\Omega)$, one can use Poincaré's inequality

$$\int_\Omega u^2 \leq C_\Omega \int_\Omega |\nabla u|^2 \tag{9}$$

where $C_\Omega \geq 0$ only depends on the domain Ω. Additionally,

$$\int_\Omega |\nabla u|^2 = -\int_\Omega u \cdot \Delta u \leq \int_\Omega |u| \cdot |\Delta u|$$

Since, by the definition of $H_0^2(\Omega)$ (using smooth approximation by smooth compactly supported functions), the integration by parts formula used in the first equality above holds. Further, using a weighted Cauchy-Schwarz inequality with $\epsilon > 0$ pointwise on the integrand above

$$\int_\Omega |u| \cdot |\Delta u| \leq \epsilon \int_\Omega u^2 + \frac{C}{\epsilon} \int_\Omega (\Delta u)^2$$

Now, combining the above yields

$$\int_\Omega u^2 \leq C_\Omega \cdot \epsilon \int_\Omega u^2 \; + \frac{C \cdot C_\Omega}{\epsilon} \int_\Omega (\Delta u)^2$$

Choose $\epsilon = \tilde{\epsilon} = \frac{1}{2C_\Omega + 1} > 0$ which gives

$$\frac{1}{2} \int_\Omega u^2 \leq (1 - C_\Omega \tilde{\epsilon}) \int_\Omega u^2 \leq \frac{C \cdot C_\Omega}{\epsilon} \int_\Omega (\Delta u)^2$$

Thus,

$$\int_\Omega u^2 \leq \frac{2C \cdot C_\Omega}{\epsilon} \int_\Omega (\Delta u)^2 \tag{10}$$

from which we can similarly bound

$$\int_\Omega |\nabla u|^2 \leq \int_\Omega |u| \cdot |\Delta u| \leq \frac{1}{2} \int_\Omega |u|^2 + \frac{1}{2} \int_\Omega |\Delta u|^2$$

$$\leq \left[\frac{C \cdot C_\Omega}{\epsilon} + \frac{1}{2} \right] \int_\Omega (\Delta u)^2 \tag{11}$$

By another application of Cauchy-Schwarz and of integration by parts. Now for $u \in C_0^\infty(\Omega)$, repeated integration by parts yields

$$\int_\Omega \partial_{ii} u \partial_{jj} u = - \int_\Omega \partial_{iij} u \partial_j u = \int_\Omega \partial_{ij} u \partial_{ij} u$$

Now, by smooth approximation, any $u \in H_0^2(\Omega)$ satisfies

$$\int_\Omega \partial_{ii} u \partial_{jj} u = \int_\Omega \partial_{ij} u \partial_{ij} u$$

Thus,

$$\int_\Omega \sum_{1 \leq i,j \leq n} \left(\partial_{ij} u \right) \left(\partial_{ij} u \right) = \int_\Omega \sum_{1 \leq i,j \leq n} (\partial_{ii} u) \left(\partial_{jj} u \right) = \int_\Omega (\Delta u)^2 \tag{12}$$

Combining (10), (11) and (12) one obtains

$$\|u\|^2_{H_0^2(\Omega)} = \int_\Omega u^2 + \int_\Omega |\nabla u|^2 + \int_\Omega \sum_{1 \leq i,j \leq n} \left(\partial_{ij} u \right) \left(\partial_{ij} u \right)$$

$$\leq \frac{2C \cdot C_\Omega}{\epsilon} \int_\Omega (\Delta u)^2 + \left[\frac{C \cdot C_\Omega}{\epsilon} + \frac{1}{2} \right] \int_\Omega (\Delta u)^2 + \int_\Omega (\Delta u)^2$$

and finally,

$$\|u\|^2_{H_0^2(\Omega)} \leq \left[\frac{C \cdot C_\Omega}{\epsilon} + \frac{3}{2} \right] \int_\Omega (\Delta u)^2 \tag{13}$$

Thus, for

$$\beta = \frac{1}{\frac{C \cdot C_\Omega}{\epsilon} + \frac{3}{2}} > 0$$

one has that

$$B[u,u] \geq \beta \cdot \|u\|^2_{H^2_0(\Omega)}$$

showing coercivity. To complete the proof of existence and uniqueness, it is indeed that case that the linear functional

$$v \mapsto \int_\Omega f v, \quad v \in H^2_0(\Omega)$$

is bounded. This is easily seen by applying Cauchy Schwarz yielding

$$\left|\int_\Omega f v\right| \leq \int_\Omega |f||v| \leq \|f\|_2 \cdot \|v\|_{H^2_0(\Omega)}$$

thereby enabling the use of Lax-Milgram, showing existence and uniqueness of weak solutions finishing the proof.

4. Consider the function

$$\psi(x) = \exp[-(a+ib)|x|^2], \quad x \in \mathbb{R}^d, a > 0, b \in \mathbb{R}.$$

Its L^p norm $(1 \leq p \leq \infty, p \neq 2)$is computed using the standard formula for the integral of a Gaussian:

$$\|\psi\|_p = \begin{cases} \left(\int_{\mathbb{R}^d} |\psi|^p\right)^{\frac{1}{p}} = \left(\int_{\mathbb{R}^d} \exp\left(-ap|x|^2\right)\right)^{\frac{1}{p}} = \left(\frac{\pi}{ap}\right)^{\frac{d}{2p}}, p < \infty \\ \operatorname*{ess\,sup}_{\mathbb{R}^d} |\psi| = \sup_{\mathbb{R}^d} |\psi| = 1, p = \infty \end{cases} \tag{14}$$

Its Fourier transform is computed

$$\begin{aligned} \mathcal{F}(\psi)(\xi) &= \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} \psi(x) dx = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} \exp[-(a+ib)|x|^2] dx \\ &= \frac{1}{(2\pi)^{\frac{d}{2}}} \exp\left[-\frac{|\xi|^2}{4(a+ib)}\right] \int_{\mathbb{R}^d} \exp\left[-(a+ib)\left|x + i\frac{\xi}{2(a+ib)}\right|^2\right] dx \\ &= \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\frac{\pi}{(a+ib)}\right)^{\frac{d}{2}} \exp\left[-\frac{|\xi|^2}{4(a+ib)}\right], \quad \xi \in \mathbb{R}^d \end{aligned} \tag{15}$$

since $a > 0$ and using the standard formula for the value of the integral of a Gaussian again in the final line. Now, from lectures, the solution map to the PDE

$$\partial_t u = i\Delta u, \quad u(0,\cdot) = \psi \in \mathcal{S}(\mathbb{R}^d) \tag{16}$$

is given by:

$$S(t)\psi(x) = \mathcal{F}^{-1}\left[e^{-it|\xi|^2} \mathcal{F}(\psi)(\xi)\right](x), \quad t \in \mathbb{R}, x \in \mathbb{R}^d$$

More explicitly, using (15):

$$S(t)f(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-it|\xi|^2 + ix\cdot\xi} \mathcal{F}(\psi)(\xi) d\xi$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-it|\xi|^2 + ix\cdot\xi} \left(\frac{\pi}{(a+ib)}\right)^{\frac{d}{2}} \exp\left[-\frac{|\xi|^2}{4(a+ib)}\right] d\xi$$

$$= \left(\frac{\pi}{(a+ib)}\right)^{\frac{d}{2}} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \exp\left[-\left(\frac{1}{4(a+ib)} + it\right)|\xi|^2\right] d\xi$$

$$= \left(\frac{\pi}{(a+ib)}\right)^{\frac{d}{2}} \frac{1}{(2\pi)^{\frac{d}{2}}} \mathcal{F}\left(\exp\left[-\left(\frac{1}{4(a+ib)} + it\right)|\xi|^2\right]\right)(-x)$$

$$= \left(\frac{\pi}{(a+ib)}\right)^{\frac{d}{2}} \cdot \frac{1}{(2\pi)^d} \left(\frac{\pi}{\left(\frac{1}{4(a+ib)} + it\right)}\right)^{\frac{d}{2}} \exp\left[-\frac{|x|^2}{\left(\frac{1}{(a+ib)} + i4t\right)}\right] \tag{17}$$

where (15) was used in the last line in conjunction with the fact that $Re\left\{\frac{1}{4(a+ib)}\right\} = \frac{a}{4(a^2+b^2)} > 0$. Computing the L^p norm for $p < \infty$:

$$\|S(t)\psi(x)\|_p = \left(\int_{\mathbb{R}^d} |S(t)\psi(x)|^p\right)^{\frac{1}{p}}$$

$$= \left(\frac{\pi}{(a^2+b^2)^{\frac{1}{2}}}\right)^{\frac{d}{2}} \cdot \frac{1}{(2\pi)^d} \left(\frac{\pi}{\left|\frac{1}{4(a+ib)} + it\right|}\right)^{\frac{d}{2}} \left(\int_{\mathbb{R}^d} \exp\left[-\frac{ap(a^2+b^2)}{a^2+(b-4t(a^2+b^2))^2}|x|^2\right]\right)^{\frac{1}{p}}$$

$$= \left(\frac{\pi}{(a^2+b^2)^{\frac{1}{2}}}\right)^{\frac{d}{2}} \cdot \frac{1}{(2\pi)^d} \left(\frac{\pi}{\left|\frac{1}{4(a+ib)} + it\right|}\right)^{\frac{d}{2}} \cdot \left(\frac{\pi(a^2+(b-4t(a^2+b^2))^2)}{ap(a^2+b^2)}\right)^{\frac{d}{2p}} \tag{18}$$

Now, combining (14) and (18), one computes:

$$\frac{\|S(t)\psi\|_p}{\|\psi\|_p} = \left(\frac{\pi}{(a^2+b^2)^{\frac{1}{2}}}\right)^{\frac{d}{2}} \cdot \frac{1}{(2\pi)^d} \left(\frac{\pi}{\left|\frac{1}{4(a+ib)} + it\right|}\right)^{\frac{d}{2}} \cdot \left(\frac{\pi(a^2+(b-4t(a^2+b^2))^2)}{ap(a^2+b^2)}\right)^{\frac{d}{2p}} \Big/ \left(\frac{\pi}{ap}\right)^{\frac{d}{2p}}$$

$$= \frac{1}{(2\pi)^d} \left(\frac{\pi}{(a^2+b^2)^{\frac{1}{2}}}\right)^{\frac{d}{2}} \cdot \left(\frac{4\pi(a^2+b^2)}{|a-ib+i4t(a^2+b^2)|}\right)^{\frac{d}{2}} \cdot \left(\frac{(a^2+(b-4t(a^2+b^2))^2)}{(a^2+b^2)}\right)^{\frac{d}{2p}}$$

$$= \left(\frac{(a^2+b^2)^{\frac{1}{2}}}{|a-ib+i4t(a^2+b^2)|}\right)^{\frac{d}{2}} \cdot \left(\frac{|a-ib+i4t(a^2+b^2)|}{(a^2+b^2)^{\frac{1}{2}}}\right)^{\frac{d}{p}} = \left(\frac{|a-ib+i4t(a^2+b^2)|}{(a^2+b^2)^{\frac{1}{2}}}\right)^{\left(\frac{1}{p}-\frac{1}{2}\right)d} \tag{19}$$

Now, for $1 \leq p < 2$, notice that

$$\alpha = \frac{1}{p} - \frac{1}{2} > 0$$

Consider the sequence $(\psi_n)_{n\geq 0}$ given by:

$$\psi_n(x) = \exp\left[-n|x|^2\right] \neq 0 \in \mathcal{S}(\mathbb{R}^d)$$

Then, using (19), one computes for $p < \infty$ and $b = 0$, $t \neq 0$:

$$\frac{\|S(t)\psi\|_p}{\|\psi\|_p} = \left(\frac{\left|a + i4ta^2\right|}{a}\right)^{\alpha d} = |1 + i4ta|^{\alpha d} \tag{20}$$

One computes using (20) for $t \neq 0$:

$$\frac{\|S(t)\psi_n\|_p}{\|\psi_n\|_p} = (|1 + i4tn|^{\alpha})^d \sim n^{\alpha d} \to \infty, \quad \text{as} \quad n \to \infty.$$

Now for the case where $p \in (2, \infty), t \neq 0$, notice

$$\alpha = \frac{1}{p} - \frac{1}{2} < 0$$

Choose $(\psi_n)_{n\geq 0}$ given by:

$$\psi_n(x) = \exp\left[-\left(\frac{1}{n} + \frac{1}{4t}\right)|x|^2\right] \neq 0 \in \mathcal{S}(\mathbb{R}^d)$$

and compute using (19) with $a = \frac{1}{n}, b = \frac{1}{4t}$:

$$\frac{\|S(t)\psi_n\|_p}{\|\psi_n\|_p} = \left(\frac{\left|a - ib + i4t(a^2 + b^2)\right|}{(a^2 + b^2)^{\frac{1}{2}}}\right)^{\left(\frac{1}{p} - \frac{1}{2}\right)d} = \left(\left|\frac{1}{a + ib} + i4t\right|(a^2 + b^2)^{\frac{1}{2}}\right)^{\left(\frac{1}{p} - \frac{1}{2}\right)d}$$

$$= \left(\left|\frac{1}{\frac{1}{n} + ib} + ib\right|\left(\frac{1}{n^2} + b^2\right)^{\frac{1}{2}}\right)^{\left(\frac{1}{p} - \frac{1}{2}\right)d}$$

But, as $n \to \infty$,

$$\left|\frac{1}{\frac{1}{n} + ib} + i4t\right| \to 0, \quad \text{and} \quad \left(\frac{1}{n^2} + b^2\right)^{\frac{1}{2}} \to \left|\frac{1}{4t}\right| \tag{21}$$

Thus,

$$\frac{\|S(t)\psi_n\|_p}{\|\psi_n\|_p} \to \infty, \quad n \to \infty$$

since $\alpha d < 0$ For the final case of $p = \infty$, using (14) and (17):

$$\frac{\|S(t)\psi\|_\infty}{\|\psi\|_\infty} = \|S(t)\psi\|_\infty = \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} |S(t)\psi|$$

$$= \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \left| \left(\frac{\pi}{(a+ib)}\right)^{\frac{d}{2}} \cdot \frac{1}{(2\pi)^d} \left(\frac{\pi}{\left(\frac{1}{4(a+ib)} + it\right)}\right)^{\frac{d}{2}} \exp\left[-\frac{|x|^2}{\left(\frac{1}{(a+ib)} + i4t\right)}\right]\right|$$

$$= \left(\frac{\pi}{|a+ib|}\right)^{\frac{d}{2}} \cdot \frac{1}{(2\pi)^d} \left(\frac{\pi}{\left|\frac{1}{4(a+ib)} + it\right|}\right)^{\frac{d}{2}} \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \left|\exp\left[-\frac{|x|^2}{\left(\frac{1}{(a+ib)} + i4t\right)}\right]\right|$$

$$= \left(\frac{(a^2+b^2)^{\frac{1}{2}}}{\left|a - ib + i4t(a^2+b^2)\right|}\right)^{\frac{d}{2}} \sup_{x\in\mathbb{R}^d} \left|\exp\left[-\frac{a(a^2+b^2)}{a^2 + (b - 4t(a^2+b^2))^2}|x|^2\right]\right|$$

$$= \left(\frac{(a^2+b^2)^{\frac{1}{2}}}{\left|a - ib + i4t(a^2+b^2)\right|}\right)^{\frac{d}{2}} \tag{22}$$

Now, consider the sequence $(\psi_n)_{n\geq 0}$ given by:

$$\psi_n(x) = \exp\left[-\left(\frac{1}{n} + \frac{1}{4t}\right)|x|^2\right] \neq 0 \in \mathcal{S}(\mathbb{R}^d)$$

One computes using (22):

$$\frac{\|S(t)\psi_n\|_\infty}{\|\psi_n\|_\infty} = \left(\left|\frac{1}{\frac{1}{n} + ib} + i4t\right| \left(\frac{1}{n^2} + b^2\right)^{\frac{1}{2}}\right)^{-\frac{d}{2}} \to \infty, \quad \text{as} \quad n \to \infty.$$

for the same reason as in (21) and since $-\frac{d}{2} < 0$. Thus, we have showed that for all $1 \leq p \leq \infty, p \neq 2, t \neq 0$:

$$\sup\left\{\frac{\|\mathcal{S}(t)\psi\|_{L^p}}{\|\psi\|_{L^p}} : \psi \in \mathcal{S}(\mathbb{R})^d, \psi \neq 0\right\} = \infty$$

meaning that $\mathcal{S}(t)$ cannot be extended to all of $L^p(\mathbb{R}^d)$, as required.

Question 5

It suffices to show that $\frac{d}{dt}E(t) = 0$ for all $t \in \mathbb{R}$. Rewriting the energy as

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^d} \nabla u^T \overline{\nabla u} + \frac{1}{4}\int_{\mathbb{R}^d} u^2\overline{u}^2$$

and differentiating, one obtains

$$\begin{aligned}\frac{d}{dt}E(t) &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} \nabla u^T \overline{\nabla u} + \frac{1}{4}\frac{d}{dt}\int_{\mathbb{R}^d} u^2\overline{u}^2 \\ &= \frac{1}{2}\int_{\mathbb{R}^d} \frac{d}{dt}(\nabla u^T \overline{\nabla u}) + \frac{1}{4}\int_{\mathbb{R}^d} \frac{d}{dt}(u^2\overline{u}^2)\end{aligned}$$

exchanging time and space derivatives, and using the product rule one arrives at (keeping in mind that $u(t,\cdot), u_t(t,\cdot) \in \mathcal{S}(\mathbb{R}^d)$ for all $t \in \mathbb{R}$):

$$\frac{d}{dt}E(t) = \frac{1}{2}\int_{\mathbb{R}^d} (\nabla u_t^T \overline{\nabla u} + \nabla u^T \overline{\nabla u_t}) + \frac{1}{4}\int_{\mathbb{R}^d} (2uu_t\overline{u}^2 + 2u^2\overline{u}\cdot\overline{u_t}) \tag{23}$$

Now, since u satisfies

$$i\partial_t u + \Delta u = u|u|^2 \tag{24}$$

taking complex conjugates and using distributivity, yields

$$-i\partial_t\overline{u} + \Delta\overline{u} = \overline{u}\cdot|\overline{u}|^2 \tag{25}$$

Using integration by parts on the first integral in 23 , one obtains

$$\frac{d}{dt}E(t) = -\frac{1}{2}\int_{\mathbb{R}^d} (u_t\overline{\Delta u} + \overline{u_t}\Delta u) + \frac{1}{4}\int_{\mathbb{R}^d} (2uu_t\overline{u}^2 + 2u^2\overline{u}\cdot\overline{u_t})$$

Substituting the functional forms of $\Delta u, \overline{\Delta u}$ from 24, 25 results in

$$\begin{aligned}\frac{d}{dt}E(t) &= -\frac{1}{2}\int_{\mathbb{R}^d} [u_t(\overline{u}\cdot|\overline{u}|^2 + i\partial_t\overline{u}) + \overline{u_t}(u|u|^2 - i\partial_t u)) + \frac{1}{4}\int_{\mathbb{R}^d} (2uu_t\overline{u}^2 + 2u^2\overline{u}\cdot\overline{u_t}) \\ &= -\frac{1}{2}\int_{\mathbb{R}^d} (u_t\cdot\overline{u}\cdot|\overline{u}|^2 + iu_t\cdot\overline{u_t} + \overline{u_t}u\cdot u|u|^2 - i\overline{u_t}\cdot u_t) + \frac{1}{2}\int_{\mathbb{R}^d} (uu_t\overline{u}^2 + u^2\overline{u}\cdot\overline{u_t}) \\ &= -\frac{1}{2}\int_{\mathbb{R}^d} (u_t\cdot\overline{u}\cdot|\overline{u}|^2 + \overline{u_t}u\cdot u|u|^2) + \frac{1}{2}\int_{\mathbb{R}^d} (|u|^2\cdot u_t\overline{u} + u\cdot|u|^2\cdot\overline{u_t}) = 0\end{aligned}$$

upon noticing that $|\overline{u}| = |u| = u\cdot\overline{u}$, concluding the proof that $E(t)$ is constant in time.

References

[1] J. Rauch. *Partial Differential Equations*. Graduate Texts in Mathematics. Springer New York, 2012. pages 1