Imperial College London

COURSEWORK 1

IMPERIAL COLLEGE LONDON

DEPARTMENT OF MATHEMATICS

MATH71035
Analytic Methods In PDE

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Date: 6 November 2022

Problems

1. Let V be a Banach space and $(G_n)_{n\in\mathbb{N}}$ be a sequence of globally Lipschitz functions, such that
$$\sup_{n\in\mathbb{N}} \|G_n\|_{\text{Lip}} < \infty.$$
Let $(X_n)_{n\in\mathbb{N}}$be the sequence of unique solutions to the initial value problems
$$\begin{aligned} &\dot{X}_n(t) = G_n(X_n)(t), \quad \text{for all } t \in [0,\infty). \\ &X_n(0) = x \in V. \end{aligned}$$
for some fixed $x \in V$. Suppose that $G_n \to G$ locally uniformly as $n \to \infty$. Show then that X_n converges locally uniformly to a C^1 function $X : [0,\infty) \to V$ that solves the initial value problem
$$\begin{aligned} &\dot{X}(t) = G(X)(t), \quad \text{for all } t \in [0,\infty). \\ &X_n(0) = x \in V. \end{aligned}$$

2. Let $u : I \to \mathbb{R}$ be a continuous function where I is an interval.

   Set $\Omega = \{t \in I : u(t) \leq 2A\} \subseteq I$. Suppose that $\Omega \neq \emptyset$. Prove using the continuity property that $\Omega = I$.

   Recall that the continuity property states that for non-empty $A \subseteq I \subseteq \mathbb{R}$, where I is an interval, if A is both relatively open and closed in I, then $A = I$.

3. Suppose there is a classical (i.e. C^1) solution $u : [0,T) \times \mathbb{R} \to \mathbb{R}$, where $T \in [0,\infty]$ to
$$\partial_t u + a(u)\partial_x u = 0. \quad u(0,x) = h(x) \in C^1(\mathbb{R}) \tag{1}$$
where $a \in C^1(\mathbb{R})$. Show that it has the form
$$u(\bar{t},\bar{x}) = u(0,(\bar{x} - a(u(\bar{t},\bar{x}))\bar{t})) = h((x - a(u(\bar{t},\bar{x}))\bar{t}), \quad (\bar{t},\bar{x}) \in [0,T) \times \mathbb{R}. \tag{2}$$
Moreover, show that if $a \circ h$ is not non-decreasing, there cannot exist global classical solutions to (5).

   Recall the blow-up lemma where one considers the initial value problem
$$\frac{d}{dt}\phi = f(\phi), \quad \phi(t_0) = \phi_0$$
with f locally Lipschitz (so in particular Lipschitz on compact domains). The blow-up lemma states that if the maximal time of existence is t_{fin}, then $\lim_{t\to t_{fin}} \|\phi(t)\| = \infty$ has to hold. This means that if the solution to the above ODE only exists for $t_0 < t_{fin} < \infty$ then necessarily $\lim_{t\to t_{fin}} \|\phi(t)\| = \infty$. A similar statement holds for the past direction.

4. Let P be the partial differential operator.

$$P(x,\partial) = \sum_{|\alpha|\le m} a_\alpha(x)\partial^\alpha \tag{3}$$

where $m \in \mathbb{N}$. Show that $\Omega = \{x \in \mathbb{R}^d : P(x,\partial) \text{ is elliptic at } x\}$ is open in $\mathbb{R}^d$.

Recall the definition of ellipticity for the partial differential equation

$$P(u)(x) = \sum_{|\alpha|\le m} a_\alpha(x)\partial^\alpha u(x) = 0 \tag{4}$$

at the point $\bar{x} = (\bar{t}, \bar{x}_2, \dots, \bar{x}_d) \in \mathbb{R}^d$ is that no hypersurface containing the aforementioned point is characteristic.

5. Fix $\zeta \in \mathbb{R}^d, \zeta \neq 0$ and let $H = \{x \in \mathbb{R}^d : \langle x, \zeta\rangle \geq 0\}$. Assume that Σ is characteristic (at any point $\bar{x} \in \Sigma$).

   Construct a smooth, non-vanishing solution to the PDE

$$P(u)(x) = \sum_{|\alpha|=m} a_\alpha \partial^\alpha u(x) = 0$$

   supported on H.

Solutions

1. Claim: $G : V \to V$ is globally Lipscitz continuous with constant K.

   To see this, we invoke the uniform convergence of the G_n to G on bounded sets, in particular for singletons and obtain that for all $x \in V$, $G_n \to G$, as $n \to \infty$. Thus, by the algebra of limits

$$\begin{aligned} \lim_{n\to\infty} |G_n(x) - G_n(y)| \le K|x-y| \implies \left|\lim_{n\to\infty} (G_n(x) - G_n(y))\right| \le K|x-y| \\ \implies |G(x) - G(y)| \le K|x-y| \end{aligned}$$

   This now allows us to use the Global Picard-Lindeloff Theorem for Banach spaces to construct the unique global solution $X : [0,\infty) \to V$ to the ODE

$$\dot{X} = G(X), \quad X(0) = x$$

   Now, fix $T \in [0,\infty)$. Clearly, $[0,T]$ is bounded in $(\mathbb{R}, |\cdot|_1)$.

Also, for all $t \in [0, T]$ and $n \in \mathbb{N}$, we have the following integral representation for X and X_n (the construction of the Riemann integral in general Banach spaces mirrors that in $(\mathbb{R}, |\cdot|_1)$):

$$X(t) = x + \int_0^t G(X(s))ds$$

$$X_n(t) = x + \int_0^t G_n(X_n(s))ds$$

We define the sequence of continuously differentiable functions $(g_n)_{n\in\mathbb{N}} : [0, T] \to V$ by

$$g_n(t) = x + \int_0^t G_n(X(s))ds$$

and observe that they converge uniformly to X on $[0, T]$. This is due to the uniform convergence of the G_n to G on the bounded set $X([0, T])$, whose boundedness follows from the continuity of X on the compact interval $[0, T]$ (whose compactness follows from the Heine-Borel Theorem). More explicitly for all $t \in [0, T]$:

$$|g_n(t) - X(t)| \leq \int_0^t |G_n(X(s)) - G(X(s))|ds \leq \sup_{y\in X([0,T])} |G_n(y) - G(y)| \cdot t$$

$$\leq \sup_{y\in X([0,T])} |G_n(y) - G(y)| \cdot T \to 0 \quad \text{as} \quad n \to \infty$$

$$\implies \sup_{t\in[0,T]} |g_n(t) - X(t)| \to 0 \quad \text{as} \quad n \to \infty$$

It suffices to show that X_n converges to X uniformly on $[0, T]$. For then, given arbitrary $\tilde{K}$ compact in $[0, \infty)$, by the boundedness $\tilde{K}$ of one finds $T \in \mathbb{R}$ large enough so that $\tilde{K} \subseteq [0, T]$. To this end we now show that

$$\sup_{t\in[0,T]} |X_n(t) - X(s)| \to 0 \quad \text{as} \quad n \to \infty$$

Fix $t \in [0, T]$ arbitrary. Now, by the triangle inequality,

$$|X_n(t) - X(t)| \leq |X_n(t) - g_n(t)| + |g_n(t) - X(t)|$$

Thus,

$$\sup_{t\in[0,T]} |X_n(t) - X(t)| \leq \sup_{t\in[0,T]} |X_n(t) - g_n(t)| + \sup_{t\in[0,T]} |g_n(t) - X(t)|$$

Now, since g_n converges uniformly to X on $[0, T]$, it remains to show that

$$\sup_{t\in[0,T]} |X_n(t) - g_n(t)| \to 0 \quad \text{as} \quad n \to \infty$$

Fix again $t \in [0, T]$ and $\epsilon > 0$ arbitrary. Further, using the integral representations and definition of g_n

$$|X_n(t) - g_n(t)| = \left| \int_0^t G_n(X_n(s))ds - \int_0^t G_n(X(s))ds \right|$$

$$\leq \int_0^t |G_n(X_n(s)) - G_n(X(s))|\, ds \leq K \int_0^t |X_n(s) - X(s)|\, ds$$

$$\leq K \int_0^t |X_n(s) - g_n(s)|\, ds + K \int_0^t |g_n(s) - X(s)|\, ds$$

$$\leq \int_0^t K\, |X_n(s)) - g_n(s)|\, ds + KT \cdot \sup_{t \in [0,T]} |g_n(t) - X(t)|$$

Now we are in a position to apply Gronwall's inequality since $KT \cdot \sup_{t\in[0,T]} |g_n(t) - X(t)|$, $K \geq 0$ and $|X_n(t) - g_n(t)| \geq 0$ is continuous yielding

$$|X_n(t) - g_n(t)| \leq KT \cdot \sup_{t \in [0,T]} |g_n(t) - X(t)| \cdot \exp\left(\int_0^t K ds \right)$$

$$\leq KT \cdot \sup_{t \in [0,T]} |g_n(t) - X(t)| \cdot \exp(KT)$$

Since this bound is uniform in $t \in [0, T]$, we have that

$$\sup_{t \in [0,T]} |X_n(t) - g_n(t)| \leq KT \cdot \sup_{t \in [0,T]} |g_n(t) - X(t)| \cdot \exp(KT) \to 0 \quad \text{as} \quad n \to \infty$$

and we are done.

2. Set $\Omega = \{t \in I : u(t) \leq 2A\} \subseteq I$. To invoke the continuity property and deduce that $\Omega = I$, we need to show that Ω is non-empty, and both opened and closed in I. It is clear by construction that $\Omega \neq \emptyset$ since we assume that there is a $t_0 \in I$ such that $u(t_0) \leq 2A$.

   For closedness of Ω, we take a sequence $(t_n)_{n\in\mathbb{N}} \subseteq \Omega \to t \in I$. Using the continuity of u on I we easily obtain

$$u(t_n) \leq 2A \implies u(t) = \lim_{n\to\infty} u(t_n) \leq 2A \implies t \in \Omega$$

   Note we have not made mention of a choice of $\epsilon > 0$; this will be needed to be done below.

   For openness in I, set

$$M = \sup_{x\in[0,2A+1]} |F(x)|$$

   Since F is by construction bounded on bounded intervals, it follows that $M < \infty$. Now suppose $t \in \Omega$. By the continuity of u on I, there exists a $\delta > 0$ such that $u((t-\delta, t+\delta)\cap I) \subseteq (u(t)-1, u(t)+1)\cap[0,\infty) \subseteq [0, 2A+1]$ as u is non-negative and $u(t) \leq 2A$. Thus, we obtain that for $\epsilon = \frac{A}{M+1} > 0$ and $\bar{t} \in (t-\delta, t+\delta)\cap I$:

$$u(\bar{t}) \leq A + \epsilon F(u(\bar{t})) \implies u(\bar{t}) \leq A + M \cdot \frac{A}{M+1} \leq 2A$$

   Thus, $(t-\delta, t+\delta)\cap I \subseteq \Omega$ and we are done.

3. Suppose there is a classical (i.e. C^1) solution $u : [0,T) \times \mathbb{R} \to \mathbb{R}$, where $T \in [0,\infty]$ to

$$\partial_t u + a(u)\partial_x u = 0. \quad u(0,x) = h(x) \in C^1(\mathbb{R}) \tag{5}$$

Fix $(\bar{t},\bar{x}) \in [0,T) \times \mathbb{R}$ arbitrary. We seek characteristic curves on which the solution $u(t,x)$ is constant. This can be achieved by considering the following ODE:

$$\dot{x} = a \circ u(t,x(t)), \quad x(\bar{t}) = \bar{x}$$

Observe that since $a \circ u \in C^1([0,T) \times \mathbb{R})$, it is locally Lipschitz hence by the Picard-Lindeloff Theorem we can guarantee the existence of a unique C^1 solution $x(t)$ in a neighbourhood around $(\bar{t},\bar{x})$. Along such a solution,

$$\frac{d}{dt}u(t,x(t)) = \partial_t u + a \circ u(t,x(t))\partial_x u = 0$$

We can do even more, and show that the characteristic curves extend up to time $t = 0$. This is not hard to show since the solution

$$x(t) = \bar{x} + \int_{\bar{t}}^{t} a \circ u(s,x(s))ds$$

remains bounded in the maximal existence interval around $\bar{t}$, since the integrand is constant. This enables us to extend the maximal existence time to $t = 0$, for suppose it was less, then by the blow-up lemma, we could obtain that

$$\sup_{t\in[0,\bar{t}]} |x(t)| = \infty$$

a contradiction to the above discussion. Consequently, the line $\Gamma = \{(\bar{t},\bar{x}) + (t-\bar{t})(1,a(u(\bar{t},\bar{x})) : t \in [0,\bar{t}]\}$ is characteristic and u is constant on Γ. Hence, $u|_\Gamma = u(\bar{t},\bar{x})$. Letting $t = 0$, we obtain

$$u(\bar{t},\bar{x}) = u(0,(\bar{x} - a(u(\bar{t},\bar{x}))\bar{t})) = h((x - a(u(\bar{t},\bar{x}))\bar{t}) \tag{6}$$

as required.

Now we show that if $a \circ h$ is not non-decreasing, there cannot exist global classical solutions to (5). Suppose for contradiction there exists a solution $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$. Pick $\xi_1 < \xi_2$ arbitrary, then by construction $a \circ h(\xi_1) > a \circ h(\xi_2)$. This means the corresponding characteristics $a(h(\xi_1))t + \xi_1, a(h(\xi_2))t + \xi_2$ intersect for time $t = \frac{a(h(\xi_1))-a(h(\xi_2))}{\xi_2-\xi_1} > 0$ at $x = a(h(\xi_1))t + \xi_1 = a(h(\xi_2))t + \xi_2$. By (6), we have $u(t,x) = h(\xi_1) = h(\xi_2)$ yielding that h is globally constant. This yields that $a \circ h$ is constant too, but this cannot hold since $a \circ h$ was assumed decreasing, a contradiction. Thus there does not exist a global classical solution to (5) and we are done.

4. We need to show that $\Omega = \{x \in \mathbb{R}^d : P(x,\partial) \text{ is elliptic at } x\}$ is open in $\mathbb{R}^d$.

The definition of ellipticity for the partial differential equation

$$P(u)(x) = \sum_{|\alpha|\leq m} a_\alpha(x)\partial^\alpha u(x) = 0 \tag{7}$$

at the point $\bar{x} = (\bar{t}, \bar{x}_2, \ldots, \bar{x}_d) \in \mathbb{R}^d$ is that no hypersurface containing the aforementioned point is characteristic. Consider an arbitrary hypersurface Σ and any smooth function $\psi(x)$ such that $\psi|_\Sigma = 0$, $d\psi|_\Sigma \neq 0$ in a neighbourhood of $\bar{x}$ and (since the PDE is linear, it is equal to its own linearisation $\bar{P}$)

$$\overline{P}(\psi^m)(\bar{x}) = P(\psi^m)(\bar{x}) = \sum_{|\alpha|\leq m} a_\alpha(\bar{x})\partial^\alpha \psi^m(\bar{x}) = m! \sum_{|\alpha|=m} a_\alpha(\bar{x})(D\psi)^\alpha(\bar{x}) \neq 0$$

since Σ is assumed non-characteristic at $\bar{x}$. All terms containing derivatives of order lower than the maximal order m vanish as $\partial^\beta \psi^m(\bar{x}), \quad |\beta| < m$ contain $\psi(\bar{x})$ coefficients which vanish by assumption (can be shown by induction).

We aim to establish an equivalent characterisation of ellipticity at a point $\bar{x}$ for the partial differential operator P. This is done by restricting attention to the unit ball $D = \{\zeta \in \mathbb{R}^d : |\zeta| = 1\}$ and considering for all $\zeta \in D$ the hypersurface $\Sigma = \{x \in \mathbb{R}^d : \langle x - \bar{x}, \zeta\rangle = 0\}$ with $\psi(x) = \langle x - \bar{x}, \zeta\rangle$. Clearly, $\psi|_\Sigma = 0$, $d\psi|_\Sigma = \zeta \neq 0$ in a neighbourhood of $\bar{x}$. Thus, we obtain that

$$\overline{P}(\psi^m)(\bar{x}) = m! \sum_{|\alpha|=m} a_\alpha(\bar{x})(D\psi)^\alpha(\bar{x}) = m! \sum_{|\alpha|=m} a_\alpha(\bar{x})(\zeta)^\alpha \neq 0 \tag{8}$$

Let $f : D \to \mathbb{R}$ be given by

$$f(\zeta) = \left|\sum_{|\alpha|=m} a_\alpha(\bar{x})(\zeta)^\alpha\right|$$

Since by the above f is continuous (it is the absolute value of a polynomial in the entries of ζ), nowhere vanishing on the compact set D, it attains its infimum

$$\inf_{\zeta\in D} f(z) = \left|\sum_{|\alpha|=m} a_\alpha(\bar{x})(\bar{\zeta})^\alpha\right| > 0 \tag{9}$$

Now, suppose for a contradiction that Ω is not open at $\bar{x}$. By the above, one obtains a sequence of points $(\zeta_n)_{n\in\mathbb{N}} \subseteq D$ and $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^d \setminus \Omega$ such that $x_n \to \bar{x} \quad \text{as} \quad n \to \infty$. By assumption, (8) yields

$$\left|\sum_{|\alpha|=m} a_\alpha(x_n)(\zeta_n)^\alpha\right| = 0$$

Due to the compactness of D, we pass to a subsequence $\zeta_{n_k} \to \zeta' \in D \quad \text{as} \quad k \to \infty$. This gives that

$$0 = \left| \sum_{|\alpha|=m} a_\alpha(x_{n_k})(\zeta_{n_k})^\alpha \right| \to \left| \sum_{|\alpha|=m} a_\alpha(\bar{x})(\zeta')^\alpha \right| \geq \inf_{\zeta \in D} f(z) > 0 \quad \text{as} \quad k \to \infty$$

a contradiction to (9). Thus, we have established that there exists a neighbourhood $\mathcal{U}$ of $\bar{x}$ contained in Ω, thereby showing that Ω is open as required.

5. Fix $\zeta \in \mathbb{R}^d, \zeta \neq 0$ and let $H = \{x \in \mathbb{R}^d : \langle x, \zeta \rangle \geq 0\}$. It is clear that H has boundary $\Sigma = \{x \in \mathbb{R}^d : \langle x, \zeta \rangle = 0\}$. Since Σ is assumed characteristic (at any point $\bar{x} \in \Sigma$) we choose $\psi(x) = \langle x, \zeta \rangle$ and easily verify $\psi|_\Sigma = 0$, $d\psi|_\Sigma \neq 0$ since $D\psi = \zeta \neq 0$ (similarly to question 4 - since the PDE is linear, it is equal to its own linearisation $\bar{P}$):

$$\overline{P}(\psi^m)(\bar{x}) = P(\psi^m)(\bar{x}) = \sum_{|\alpha|=m} a_\alpha \partial^\alpha \psi^m(\bar{x}) = m! \sum_{|\alpha|=m} a_\alpha (D\psi)^\alpha(\bar{x})$$

$$= m! \sum_{|\alpha|=m} a_\alpha (\zeta)^\alpha = 0 \tag{10}$$

In the spirit of constructing a non-vanishing smooth solution to the PDE

$$P(u)(x) = \sum_{|\alpha|=m} a_\alpha \partial^\alpha u(x) = 0$$

we define the smooth function $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} e^{-\frac{1}{x}}, & x \in (0, \infty) \\ 0, & x \in (-\infty, 0] \end{cases}$$

It is a standard (for instance, it has been mentioned in first year real analysis) example of a smooth function supported on $[0, \infty)$ whose derivatives of all orders vanish at $x = 0$, yet does not vanish in a neighbourhood of 0. Define $u : \mathbb{R}^d \to \mathbb{R}$ as

$$u(x) = h(\langle x, \zeta \rangle)$$

Since h is supported on $[0, \infty)$, the support of u is contained in $\{x \in \mathbb{R}^d : \langle x, \zeta \rangle \geq 0\} = H$. Now, we claim that

$$\partial^\alpha u(x) = h^{(|\alpha|)}(\langle x, \zeta \rangle) \cdot (\zeta)^\alpha \tag{11}$$

for all $\alpha \in \mathbb{N}^d$. We proceed to prove the claim (11) by induction. The claim follow directly by the chain rule for $\alpha \in \mathbb{N}^d$ with $|\alpha| = 1$. Now for the inductive step, suppose (11) holds for $\alpha \in \mathbb{N}^d$ with $|\alpha| = k$. Now, pick $\beta \in \mathbb{N}^d$ with $|\beta| = k + 1$. Now chose $\beta_i > 0$ and let $\beta' = \beta - e_i$. Note that $|\beta'| = k$. Consider

$$\partial^\beta u(x) = \partial^i (\partial^{\beta'} u))|_x = \partial^i (h^{(|\beta'|)}(\langle x, \zeta \rangle) \cdot (\zeta)^{\beta'}) = h^{(|\beta'|+1)}(\langle x, \zeta \rangle) \cdot (\zeta)^{\beta'} \cdot \zeta^i$$

$$:= h^{(|\beta'|+1)}(\langle x, \zeta \rangle) \cdot \zeta^\beta = \partial h^{(|\beta|)}(\langle x, \zeta \rangle) \cdot (\zeta)^\beta$$

by an application of the chain rule thus completing the induction meaning (11) holds for all multi-indices $\alpha \in \mathbb{N}^d$. Hence,

$$P(u)(x) = \sum_{|\alpha|=m} a_\alpha \partial^\alpha u(x) = \sum_{|\alpha|=m} a_\alpha h^{(m)}(\langle x, \zeta \rangle) \cdot (\zeta)^\alpha = h^{(m)}(\langle x, \zeta \rangle) \cdot \sum_{|\alpha|=m} a_\alpha \cdot (\zeta)^\alpha = 0$$

This shows that u is indeed a non-zero smooth solution to the above PDE supported on H and we are done.